On Shannon Tightness of Symmetric Entropy Regions

Qi Chen
joint work with Zihan Li and Shaocheng Liu
qichen@ie.cuhk.edu.hk

School of Telecommunications Engineering
Xidian University

July 15, 2026
INC, CUHK

Given a vector $\mathbf{h}$ in the entropy space $\mathcal{H}_n \triangleq \mathbb{R}^{2^{N_n}}$, whether there exists an n-random vector $\mathbf{X}$ whose entropy function is $\mathbf{h}$?

Given a vector $\mathbf{h}$ in the entropy space $\mathcal{H}_n \triangleq \mathbb{R}^{2^{N_n}}$, whether there exists an n-random vector $\mathbf{X}$ whose entropy function is $\mathbf{h}$?

Entropy space

- $N_n = \{1, 2, \cdots, n\}$
- the entropy space $\mathcal{H}_n$ is a Euclidean space indexed by the power set of N_n.

Entropy functions

For an n-random vector $\mathbf{X} = (X_i : i \in N_n)$, its entropy function

$$\mathbf{h_X} = (H(\mathbf{X}_A) : A \subseteq N_n)$$

where $\mathbf{X}_A = (X_i : i \in A)$ is any subvector of $\mathbf{X}$.

Given a vector $\mathbf{h}$ in the entropy space $\mathcal{H}_n \triangleq \mathbb{R}^{2^{N_n}}$, whether there exists an n-random vector $\mathbf{X}$ whose entropy function is $\mathbf{h}$?

Entropy space

- $N_n = \{1, 2, \cdots, n\}$
- the entropy space $\mathcal{H}_n$ is a Euclidean space indexed by the power set of N_n.

Entropy functions

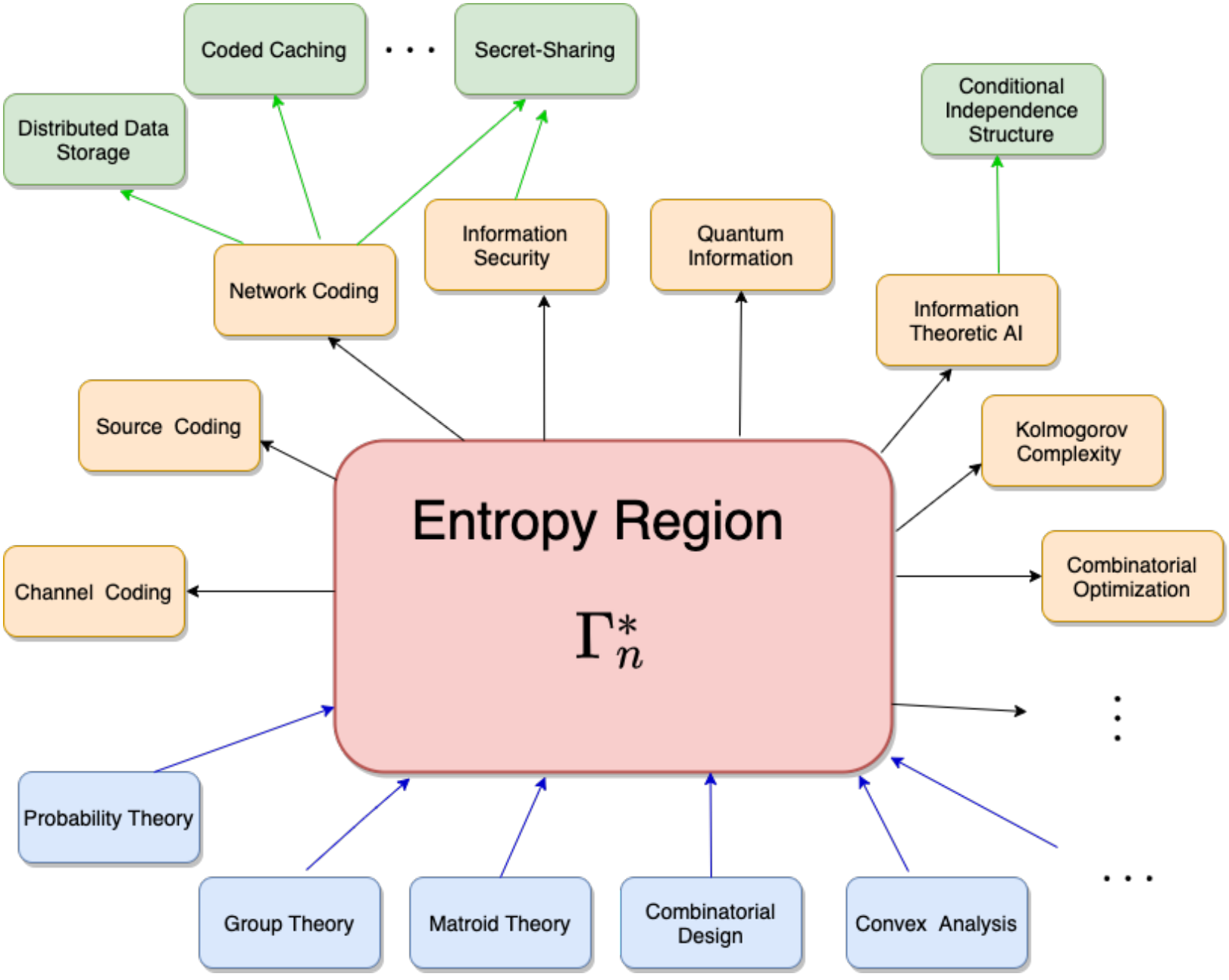
For an n-random vector $\mathbf{X} = (X_i : i \in N_n)$, its entropy function

$$\mathbf{h_X} = (H(\mathbf{X}_A) : A \subseteq N_n)$$

where $\mathbf{X}_A = (X_i : i \in A)$ is any subvector of $\mathbf{X}$.

Entropy region

$$\Gamma_n^* = \{\mathbf{h} \in \mathcal{H}_n : \exists\ \mathbf{X}, \mathbf{h} \text{ is the entropy function of some } \mathbf{X}.\}$$


Coded Caching
Secret-Sharing
Distributed Data Storage
Conditional Independence Structure
Information Security
Quantum Information
Network Coding
Information Theoretic AI
Source Coding
Kolmogorov Complexity
Entropy Region
Γ_n^*
Channel Coding
Combinatorial Optimization
Probability Theory
Group Theory
Matroid Theory
Combinatorial Design
Convex Analysis

Γ_n: Shannon outer bound on Γ_n^*

Shannon-type inequalities

For any $A, B \subseteq N_n$,

$$
\begin{aligned}
H(X_A) &\geq 0, \\
H(X_A) &\leq H(X_B) \quad \text{if } A \subseteq B, \\
H(X_A) + H(X_B) &\geq H(X_{A \cap B}) + H(X_{A \cup B}).
\end{aligned}
$$

Γ_n: Shannon outer bound on Γ_n^*

Shannon-type inequalities

For any $A, B \subseteq N_n$,

$$\begin{aligned} H(X_A) &\geq 0, \\ H(X_A) &\leq H(X_B) \quad \text{if } A \subseteq B, \\ H(X_A) + H(X_B) &\geq H(X_{A\cap B}) + H(X_{A\cup B}). \end{aligned}$$

Polymatroidal region Γ_n

$$\begin{aligned} \Gamma_n \triangleq \{\mathbf{h} \in \mathcal{H}_n : \; & \mathbf{h}(A) \geq 0, \\ & \mathbf{h}(A) \leq \mathbf{h}(B), \quad \text{if } A \subseteq B, \\ & \mathbf{h}(A) + \mathbf{h}(B) \geq \mathbf{h}(A \cap B) + \mathbf{h}(A \cup B).\} \end{aligned}$$

Γ_n^* and Γ_n

Relations between Γ_n^* and Γ_n

- $\Gamma_n^* \subseteq \Gamma_n$

Γ_n^* and Γ_n

Relations between Γ_n^* and Γ_n

- $\Gamma_n^* \subseteq \Gamma_n$
- $\Gamma_1^* = \Gamma_1$, $\Gamma_2^* = \Gamma_2$

Γ_n^* and Γ_n

Relations between Γ_n^* and Γ_n

- $\Gamma_n^* \subseteq \Gamma_n$
- $\Gamma_1^* = \Gamma_1$, $\Gamma_2^* = \Gamma_2$
- $\Gamma_3^* \subsetneq \Gamma_3$, but $\overline{\Gamma_3^*} = \Gamma_3$

Γ_n^* and Γ_n

Relations between Γ_n^* and Γ_n

- $\Gamma_n^* \subseteq \Gamma_n$
- $\Gamma_1^* = \Gamma_1$, $\Gamma_2^* = \Gamma_2$
- $\Gamma_3^* \subsetneq \Gamma_3$, but $\overline{\Gamma_3^*} = \Gamma_3$
- $\overline{\Gamma_n^*} \subsetneq \Gamma_n$, $n \geq 4$, due to the existence of non-Shannon-type information inequalities.

Γ_n^* and Γ_n

Relations between Γ_n^* and Γ_n

- $\Gamma_n^* \subseteq \Gamma_n$
- $\Gamma_1^* = \Gamma_1$, $\Gamma_2^* = \Gamma_2$
- $\Gamma_3^* \subsetneq \Gamma_3$, but $\overline{\Gamma_3^*} = \Gamma_3$
- $\overline{\Gamma_n^*} \subsetneq \Gamma_n$, $n \geq 4$, due to the existence of non-Shannon-type information inequalities.

Non-Shannon-type Information inequalities

- (Zhang-Yeung inequality, 1998) Given random variables X_1, X_2, X_3 and X_4,

$$2I(X_3; X_4) \leq I(X_1; X_2) + I(X_1; X_3, X_4) + 3I(X_3; X_4|X_1) + I(X_3; X_4|X_2).$$

Γ_n^* and Γ_n

Relations between Γ_n^* and Γ_n

- $\Gamma_n^* \subseteq \Gamma_n$
- $\Gamma_1^* = \Gamma_1$, $\Gamma_2^* = \Gamma_2$
- $\Gamma_3^* \subsetneq \Gamma_3$, but $\overline{\Gamma_3^*} = \Gamma_3$
- $\overline{\Gamma_n^*} \subsetneq \Gamma_n$, $n \geq 4$, due to the existence of non-Shannon-type information inequalities.

Non-Shannon-type Information inequalities

- (Zhang-Yeung inequality, 1998) Given random variables X_1, X_2, X_3 and X_4,

$$2I(X_3; X_4) \leq I(X_1; X_2) + I(X_1; X_3, X_4) + 3I(X_3; X_4|X_1) + I(X_3; X_4|X_2).$$

- (F. Matúš, 2007) infinitly many information inequalities

Permutation groups

- Bijection $\sigma : N_n \to N_n$ is called a permutation of degree n.
- the set of all permutations on N_n form a group S_n which is called symmetric group
- any $G \leq S_n$ is called a permutation group of degree n.

Group action

For a group G and a set Ω, a mapping $G \times \Omega \to \Omega$ is called a group action if

- $eh = h$ for the identity $e \in G$ and all $h \in \Omega$ and
- $(g_1 g_2)h = g_1(g_2 h)$ for all $g_1, g_2 \in G$ and $h \in \Omega$.

Group action

For a group G and a set Ω, a mapping $G \times \Omega \to \Omega$ is called a group action if

- $eh = h$ for the identity $e \in G$ and all $h \in \Omega$ and
- $(g_1 g_2)h = g_1(g_2 h)$ for all $g_1, g_2 \in G$ and $h \in \Omega$.

Orbits of a group action

- For any $h \in \Omega$,

$$\mathcal{O}(h) = \{gh \in \Omega : \ g \in G\}$$

  is called an orbit of the action.
- all orbits of a group action form a partition of Ω
- a group action is called transitive if it has a unique orbit, o.w. it is called intransitive

Permutation groups acting on N_n

Each permutation group $G \leq S_n$, $G \times N_n \to N_n$ with $\sigma i = \sigma(i)$ forms a group action.

Permutation groups acting on N_n

Each permutation group $G \leq S_n$, $G \times N_n \to N_n$ with $\sigma i = \sigma(i)$ forms a group action.

Example

Let $G = S_4$ and $\Omega = N_4$, then N_4 itself forms the unique orbit of the action, so S_4 is a transitive permutation group.

Permutation groups acting on N_n

Each permutation group $G \leq S_n$, $G \times N_n \to N_n$ with $\sigma i = \sigma(i)$ forms a group action.

Example

Let $G = S_4$ and $\Omega = N_4$, then N_4 itself forms the unique orbit of the action, so S_4 is a transitive permutation group.

Example

Let $G = S_2 \times S_2 = \{(1), (12), (34), (12)(34)\}$ and $\Omega = N_4$, then $\{1, 2\}$ and $\{3, 4\}$ are two orbits of the action, so $S_2 \times S_2$ is an intransitive permutation group.

Permutation groups acting on 2^{N_n}

Each permutation group $G \leq S_n$, $G \times 2^{N_n} \to 2^{N_n}$ with $\sigma A = \sigma(A) = \{\sigma(i), i \in N_n\}$ forms a group action.

Permutation groups acting on 2^{N_n}

Each permutation group $G \leq S_n$, $G \times 2^{N_n} \to 2^{N_n}$ with $\sigma A = \sigma(A) = \{\sigma(i), i \in N_n\}$ forms a group action.

Example

Let $G = S_4$ and $\Omega = 2_4^N$, then $\mathfrak{O}_G = \{\mathcal{O}(A) : |A| = k, 0 \leq k \leq 4\}$ forms the family of all permutation groups.

Permutation groups acting on 2^{N_n}

Each permutation group $G \leq S_n$, $G \times 2^{N_n} \to 2^{N_n}$ with $\sigma A = \sigma(A) = \{\sigma(i), i \in N_n\}$ forms a group action.

Example

Let $G = S_4$ and $\Omega = 2_4^N$, then $\mathfrak{O}_G = \{\mathcal{O}(A) : |A| = k, 0 \leq k \leq 4\}$ forms the family of all permutation groups.

Example

Let $G = S_2 \times S_2 = \{(1), (12), (34), (12)(34)\}$ and $\Omega = 2^{N_4}$, then

$$\begin{aligned}\mathfrak{O}_G =& \{\emptyset, \{\{1\}, \{2\}\}, \{\{3\}, \{4\}\},\\ & \{\{1,2\}\}, \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}, \{\{3,4\}\},\\ & \{\{1,2,3\}, \{1,2,4\}\}, \{\{1,3,4\}, \{2,3,4\}\},\\ & \{\{1,2,3,4\}\}\}\\ =& \{\mathcal{O}(A) : \ A \cap \{1,2\} = k_1, A \cap \{3,4\} = k_2, 0 \leq k_1, k_2 \leq 2\}\end{aligned}$$

forms the family of all permutation groups.

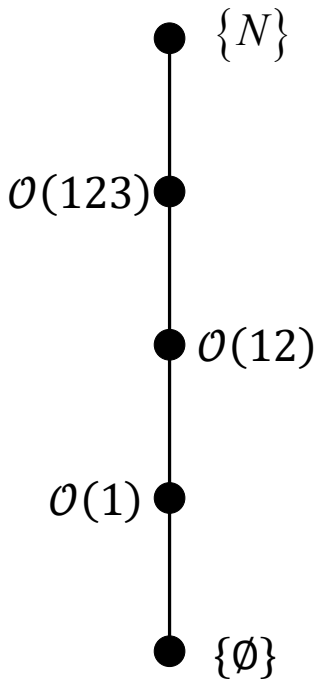
Orbit structure of G acting on 2^{N_n}

Let $\mathfrak{O}_G$ be the family of all orbits of $G \leq S_n$ acting on 2^{N_n}. For any $\mathcal{O}_1, \mathcal{O}_2$, define $\mathcal{O}_1 \leq \mathcal{O}_2$ if for any $A \in \mathcal{O}_1$, there exists $B \in \mathcal{O}_2$ such that $A \subseteq B$. Then $(\mathfrak{O}_G, \leq)$ is called the orbit structure of G.

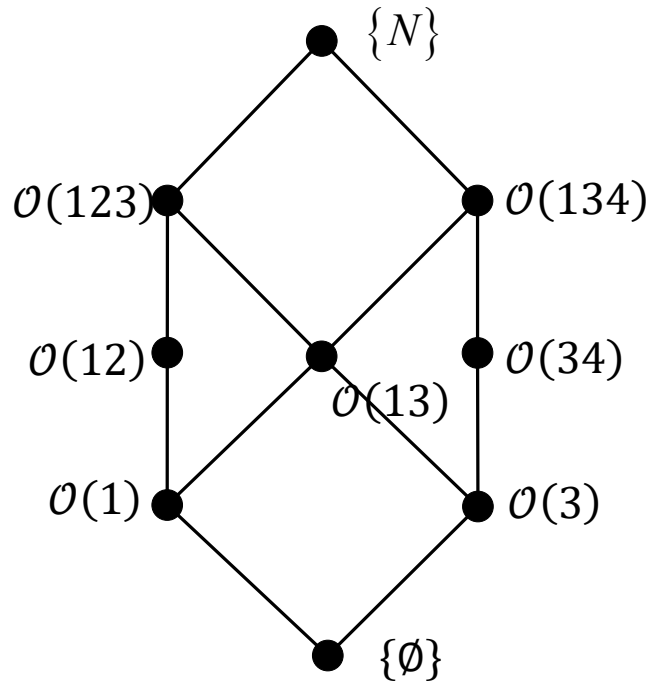
Orbit structure of G acting on 2^{N_n}

Let $\mathfrak{O}_G$ be the family of all orbits of $G \leq S_n$ acting on 2^{N_n}. For any $\mathcal{O}_1, \mathcal{O}_2$, define $\mathcal{O}_1 \leq \mathcal{O}_2$ if for any $A \in \mathcal{O}_1$, there exists $B \in \mathcal{O}_2$ such that $A \subseteq B$. Then $(\mathfrak{O}_G, \leq)$ is called the orbit structure of G.

Example



Orbit structure of S_4



Orbit structure of $S_2 \times S_2$

Symmetric entropy space

$$\begin{aligned}\mathcal{H}_G &= \{\mathbf{h} \in \mathcal{H}_n : \ \mathbf{h}(A) = \mathbf{h}(B), \text{ if } \exists \sigma \in G \text{ s.t. } \sigma(A) = B\} \\ &= \{\mathbf{h} \in \mathcal{H}_n : \ \mathbf{h}(A) = \mathbf{h}(B), B \in \mathcal{O}(A)\} \\ &= \mathbb{R}^{\mathfrak{O}_G}\end{aligned}$$

Symmetric entropy space

$$\begin{aligned}\mathcal{H}_G &= \{\mathbf{h} \in \mathcal{H}_n : \ \mathbf{h}(A) = \mathbf{h}(B), \text{ if } \exists \sigma \in G \text{ s.t. } \sigma(A) = B\} \\ &= \{\mathbf{h} \in \mathcal{H}_n : \ \mathbf{h}(A) = \mathbf{h}(B), B \in \mathcal{O}(A)\} \\ &= \mathbb{R}^{\mathfrak{O}_G}\end{aligned}$$

Examples

- When G is the trivial group, $\mathcal{H}_G = \mathcal{H}_n$ with orbit structure the boolean lattice
- When $G = S_n$, $\mathcal{H}_G$ is $n+1$-dimensional ($\emptyset$ not excluded)

Symmetric entropy regions Ψ_G^* and its Shannon-tightness

G-symmetric entropy region

$$\Psi_G^* = \Gamma_n^* \cap \mathcal{H}_G$$

G-symmetric polymatroidal region

$$\Psi_G = \Gamma_n \cap \mathcal{H}_G$$

Symmetric entropy regions Ψ_G^* and its Shannon-tightness

G-symmetric entropy region

$$\Psi_G^* = \Gamma_n^* \cap \mathcal{H}_G$$

G-symmetric polymatroidal region

$$\Psi_G = \Gamma_n \cap \mathcal{H}_G$$

Shannon-tightness of Ψ_G^*

For a permutation group $G \leq S_n$, Ψ_G^* is called

- Shannon-tight if $\overline{\Psi_G^*} = \Psi_G$; and
- Shannon-not-tight o.w.

Partition-symmetric entropy regions

Permutation groups induced by a partition

Given a partition $p = \{B_1, B_2, \ldots, B_t\}$ of N_n, the partition group induced by p is defined by

$$G_p = \{\sigma \in S_n : \sigma(B_i) = B_i \ \forall i = 1, 2, \ldots, t\}$$

or equivalently

$$S_{|B_1|} \times S_{|B_2|} \times \cdots \times S_{|B_t|}.$$

[1] Q. Chen and R. W. Yeung, "Partition-Symmetrical Entropy Functions," *IEEE Trans. Inf. Theory* vol. 62, no. 10, pp.5385–5402, Oct. 2016

Partition-symmetric entropy regions

Permutation groups induced by a partition

Given a partition $p = \{B_1, B_2, \ldots, B_t\}$ of N_n, the partition group induced by p is defined by

$$G_p = \{\sigma \in S_n : \sigma(B_i) = B_i \ \forall i = 1, 2, \ldots, t\}$$

or equivalently

$$S_{|B_1|} \times S_{|B_2|} \times \cdots \times S_{|B_t|}.$$

Theorem

For a partition p,

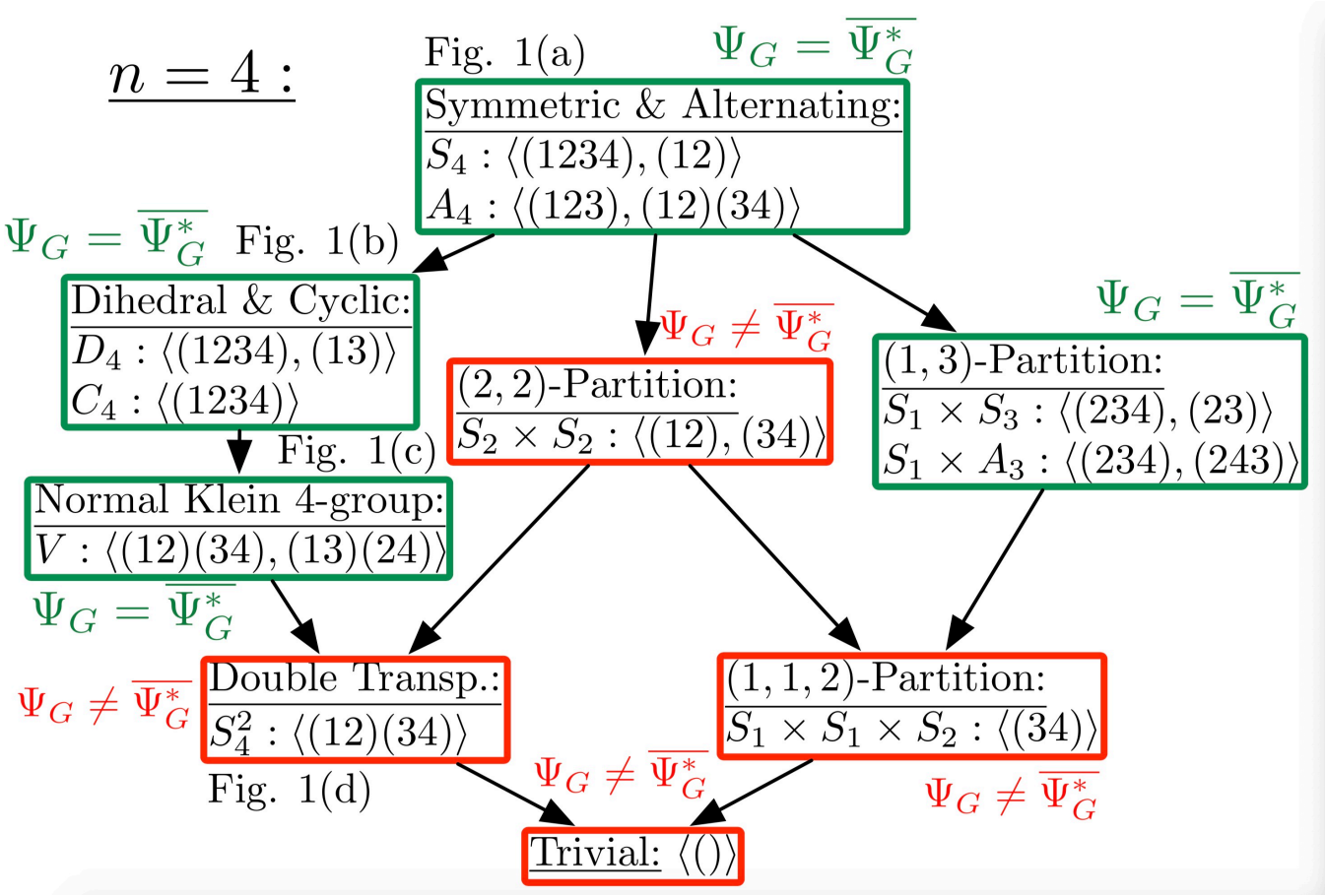
$$\overline{\Psi^*_{G_p}} = \Psi_{G_p}$$

if and only if p is a 1-partition, or a 2-partition with one of its blocks being singleton. [1]

---

[1] Q. Chen and R. W. Yeung, "Partition-Symmetrical Entropy Functions," *IEEE Trans. Inf. Theory* vol. 62, no. 10, pp.5385–5402, Oct. 2016

Symmetric-entropy functions with degree 4 [2]



[2] Apte, Q. Chen and J. W. Walsh, "Symmetries in the entropy space," IEEE Info. Theory Workshop, Cambridge, UK, Sept. 2016.

Main techniques of the proof

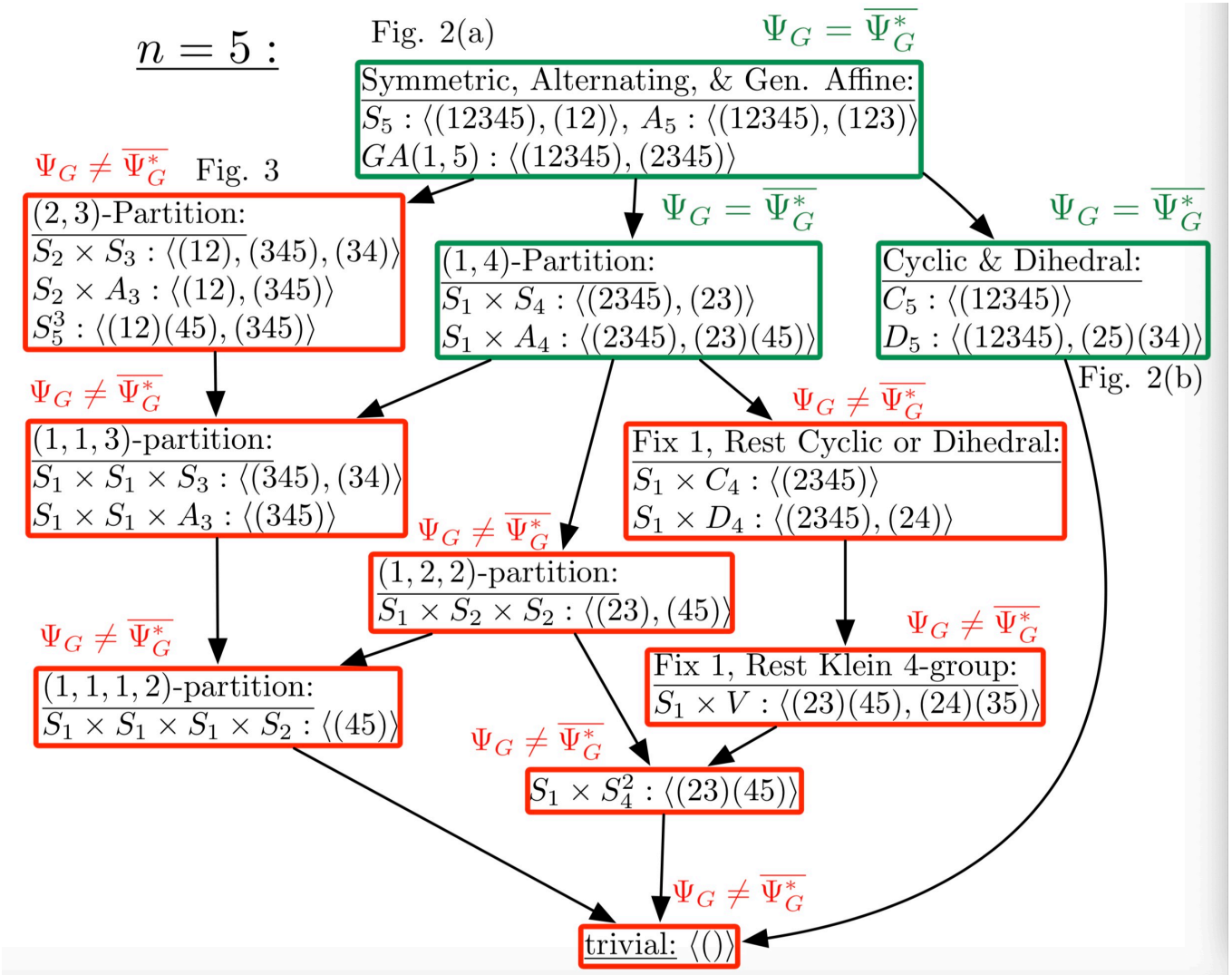
- Checking all the extreme rays of Ψ_G, if they are all almost entropic, then $\overline{\Psi^*_G} = \Psi_G$;
- if Ψ_G contains a non-entropic extreme ray violating a non-Shannon-type inequality, then $\overline{\Psi^*_G} \subsetneq \Psi_G$.

Main techniques of the proof

- Checking all the extreme rays of Ψ_G, if they are all almost entropic, then $\overline{\Psi^*_G} = \Psi_G$;
- if Ψ_G contains a non-entropic extreme ray violating a non-Shannon-type inequality, then $\overline{\Psi^*_G} \subsetneq \Psi_G$.

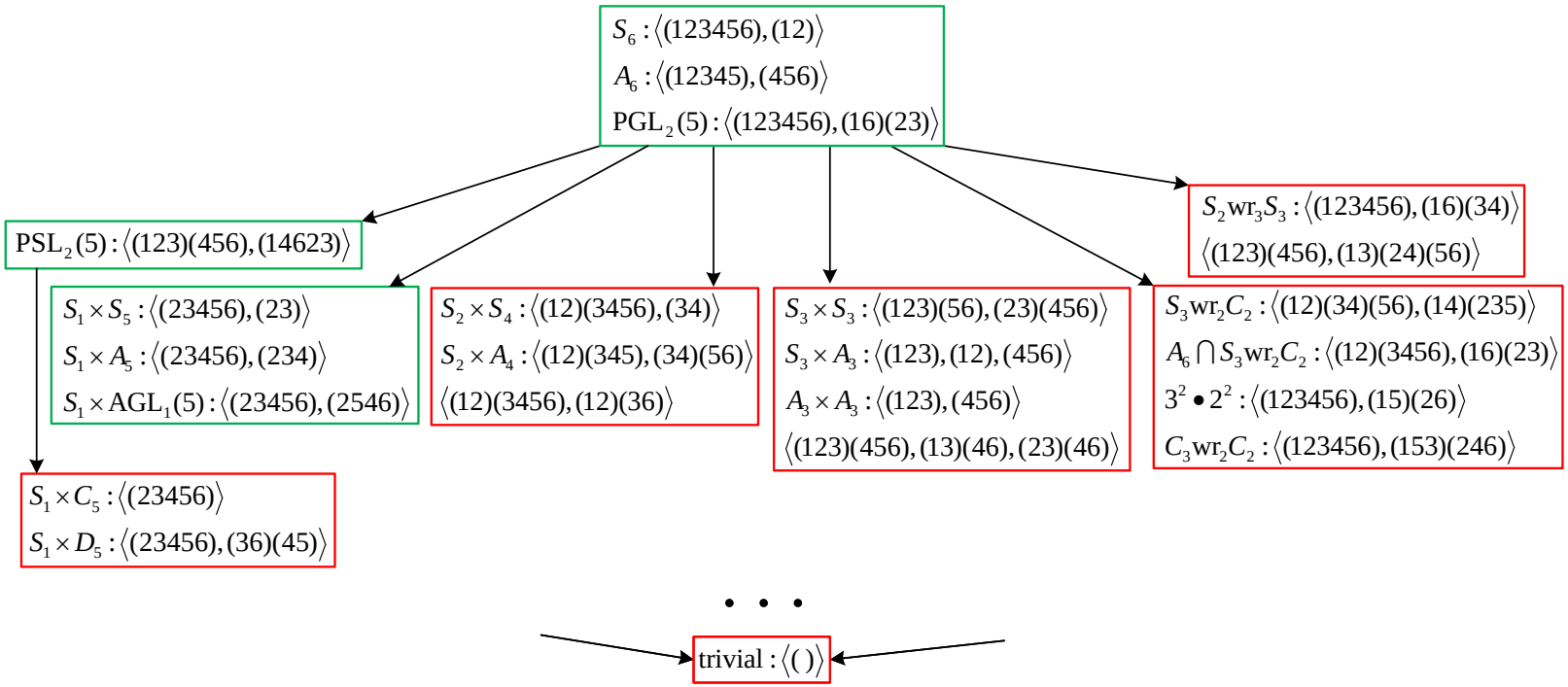
- If $G_1 \leq G_2$ and $\overline{\Psi^*_{G_1}} = \Psi_{G_1}$, then $\overline{\Psi^*_{G_2}} = \Psi_{G_2}$;
- if $G_1 \leq G_2$ and $\overline{\Psi^*_{G_2}} \subsetneq \Psi_{G_2}$, then $\overline{\Psi^*_{G_1}} \subsetneq \Psi_{G_1}$.

Symmetric-entropy functions with degree 5 [3]



[3] Apte, Q. Chen and J. W. Walsh, "Symmetries in the entropy space," IEEE Info. Theory Workshop, Cambridge, UK, Sept. 2016.

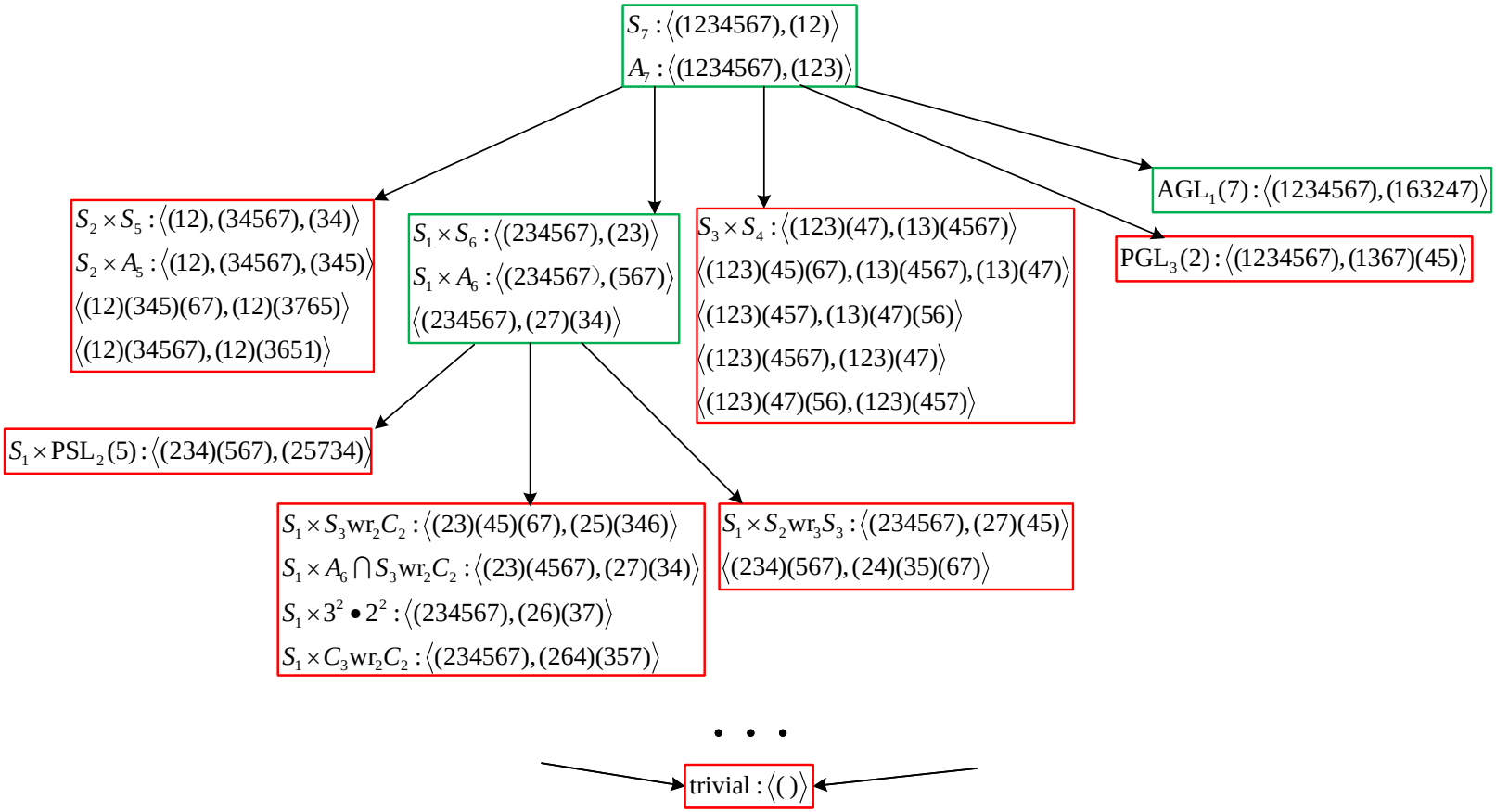
Symmetric-entropy functions with degree 6 [4]



All 35 equivalence classes of symmetric entropy regions for $n = 6$

[4] Z. Li, S. Liu and Q. Chen "Symmetric Entropy Regions of Degrees Six and Seven", IEEE Int.Symp. Info. Theory, Athens, Greece July 2024.

Symmetric-entropy functions with degree 7 [5]



All 51 equivalence classes of symmetric entropy regions for $n = 7$

[5] Z. Li, S. Liu and Q. Chen "Symmetric Entropy Regions of Degrees Six and Seven", IEEE Int.Symp. Info. Theory, Athens, Greece July 2024.

Rank structure of a polymatroid

Rank orbit

For a polymatroid P, we call $\mathcal{R}(A) = \{A : \mathbf{h}(A) = r, |A| = k\}$ with $0 \leq r \leq \mathbf{h}(N_n)$ and $0 \leq k \leq n$ a rank orbit of P.

Rank structure of a polymatroid

Rank orbit

For a polymatroid P, we call $\mathcal{R}(A) = \{A : \mathbf{h}(A) = r, |A| = k\}$ with $0 \leq r \leq \mathbf{h}(N_n)$ and $0 \leq k \leq n$ a rank orbit of P.

Rank structure

Let

$$\mathfrak{R}_P = \{\mathcal{R}(A) = \{A : \mathbf{h}(A) = r, |A| = k\}, |A| \subset N_n\}$$

be the family of all rank orbits of P. For $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}_P$, $\mathcal{R}_1 \leq \mathcal{R}_2$ if there exist $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$ such that $A \subseteq B$. Then $(\mathfrak{R}_P, \leq)$ is called the rank structure of P.

Rank structure of a polymatroid

Rank orbit

For a polymatroid P, we call $\mathcal{R}(A) = \{A : \mathbf{h}(A) = r, |A| = k\}$ with $0 \leq r \leq \mathbf{h}(N_n)$ and $0 \leq k \leq n$ a rank orbit of P.

Rank structure

Let

$$\mathfrak{R}_P = \{\mathcal{R}(A) = \{A : \mathbf{h}(A) = r, |A| = k\}, |A| \subset N_n\}$$

be the family of all rank orbits of P. For $\mathcal{R}_1, \mathcal{R}_2 \in \mathfrak{R}_P$, $\mathcal{R}_1 \leq \mathcal{R}_2$ if there exist $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$ such that $A \subseteq B$. Then $(\mathfrak{R}_P, \leq)$ is called the rank structure of P.

Lemma

The orbit structure of the automorphism group of P is a refinement of the rank structure of P.

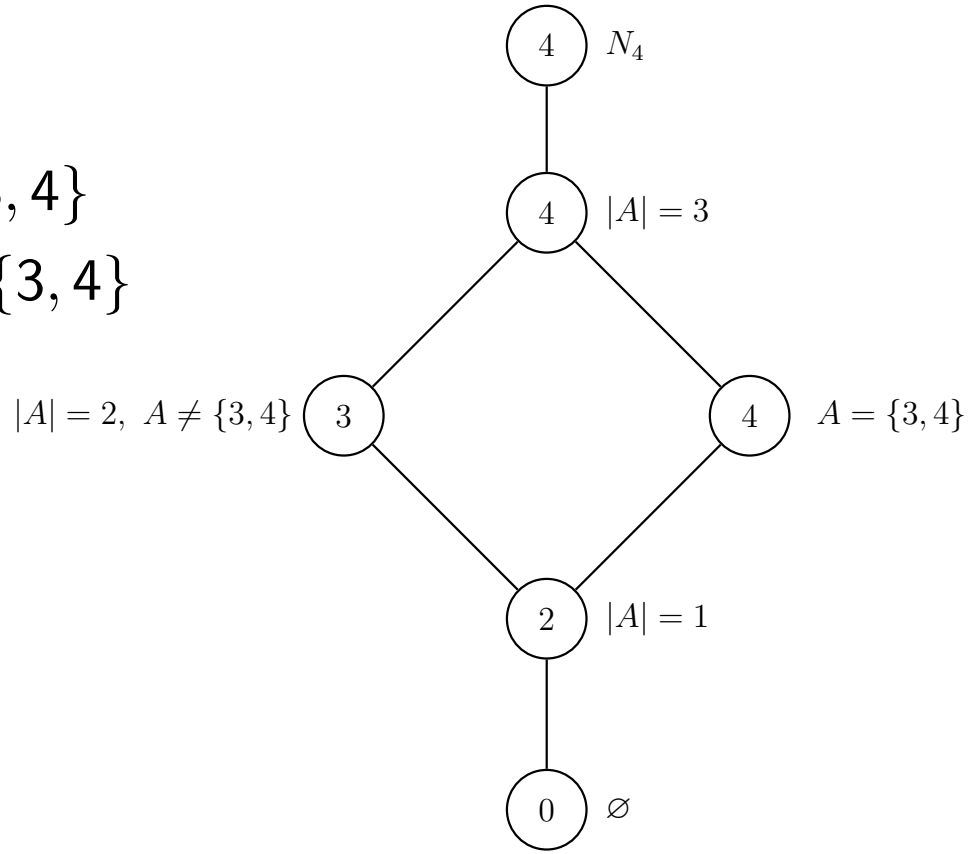
Vámos polymatroid

Polymatroid
$V^{\{3,4\}} = (N_n, \mathbf{h})$

$$\mathbf{h}(A) = \begin{cases} 4 & |A| \geq 3 \text{ or } A = \{3,4\} \\ 3 & |A| = 2 \text{ and } A \neq \{3,4\} \\ 2 & |A| = 1 \\ 0 & A = \emptyset \end{cases}$$

Its automorphism group is $S_2 \times S_2$.

Rank structure of Vámos polymatroid



Minor of a polymatroid

For a polymatroid $P = (N_4, \mathbf{h})$,

- $P' = (N_{n'}, \mathbf{h}')$ is called the restriction of P on $N_{n'}$ if $\mathbf{h}'(A) = \mathbf{h}(A)$ for all $A \subseteq N_{n'}$
- $P'' = (N_{n''}), \mathbf{h}''$ is called the contraction of P by $C = N_n \setminus N_{n''}$ if $\mathbf{h}''(A) = \mathbf{h}(A \cup C) - \mathbf{h}(C)$ for all $A \subseteq N_{n''}$

A finite sequence of restrictions and contractions on P in any order yields a polymatroid, which is called a minor of P.

Minor of an orbit structure

Let $G \leq S_n$ act on 2^{N_n} and $(\mathfrak{O}_G, \leq)$ be the orbit structure.

- For $R \subseteq N_n$, $\mathfrak{O}_G|R = \{\mathcal{O}' : \exists \mathcal{O} \in \mathfrak{O}_G \mathrm{s.t.} \mathcal{O}' \subset \mathcal{O}\}$ is called the restriction of $\mathfrak{O}_G$ on R.
- For $C \subseteq N_n$, $\mathfrak{O}_G/C = \{\mathcal{O}' : \ \mathcal{O} = \{A \cup C, A \in \mathcal{O}'\} \in \mathfrak{O}_G\}$ is called the contraction of $\mathfrak{O}_G$ by C.

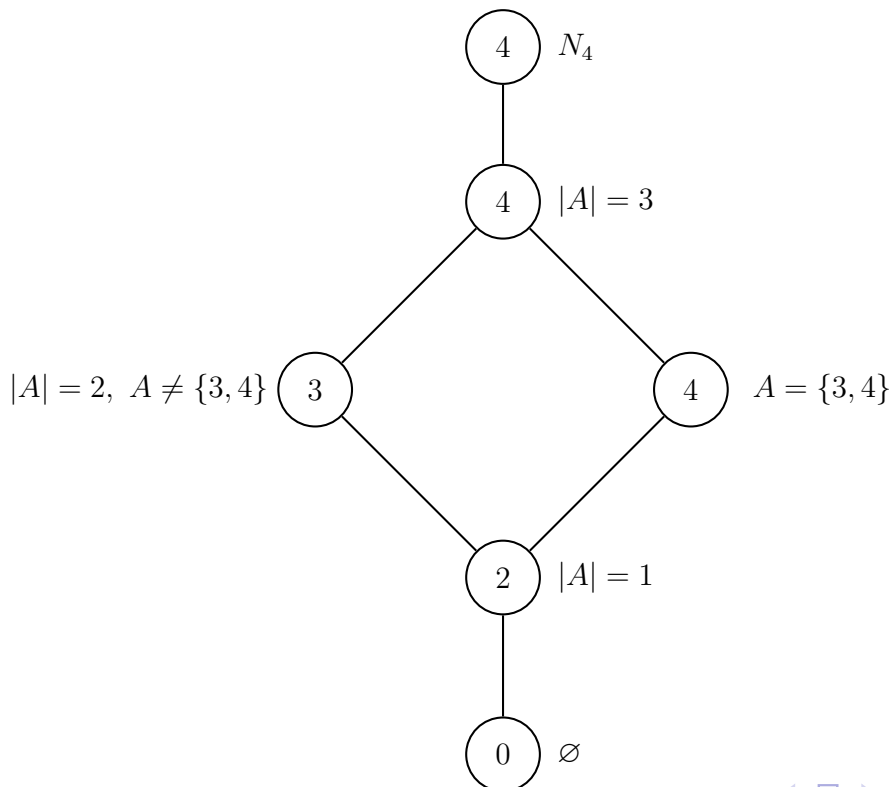
A finite sequence of restrictions and contractions on $(\mathfrak{O}_G, \leq)$ in any order yields a partial order set $(\mathfrak{O}', \leq)$, not necessarily an orbit structure, which is called a minor of $(\mathfrak{O}_G, \leq)$

Forbidden minor lemma

Lemma

For any $G \leq S_n$, $\overline{\Psi^*_G} \neq \Psi_G$ if

- *$\mathcal{O}_G$ has a minor on a 4-subset $\Omega \subseteq N_n$ which is a refinement of the rank structure of Vámos polymatroid, or more exactly,*
- *$\mathcal{O}_G$ has a minor whose family of 2-subsets orbits is a refinement of a $(1, 5)$-partition of the all 6 2-subsets of Ω.*



Examples

Partition-symmetric entropy regions

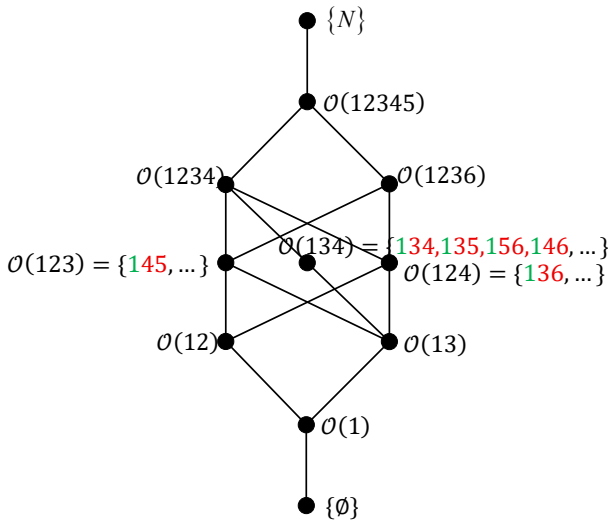
For partition group G_p with p containing at least 2 blocks with size ≥ 2, $\mathfrak{D}_{G_p}$ has a minor isomorphic to $\mathfrak{D}_{S_2 \times S_2}$ whose family of 2-subsets orbits is a $(1, 4, 1)$-partition.

Examples

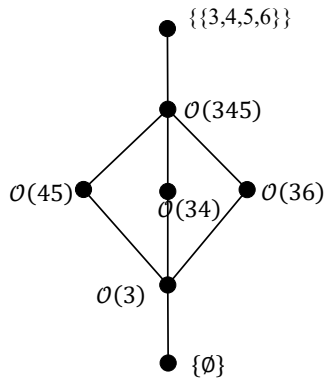
Partition-symmetric entropy regions

For partition group G_p with p containing at least 2 blocks with size ≥ 2, $\mathfrak{D}_{G_p}$ has a minor isomorphic to $\mathfrak{D}_{S_2 \times S_2}$ whose family of 2-subsets orbits is a $(1, 4, 1)$-partition.

$C_2 \mathrm{wr} C_3$



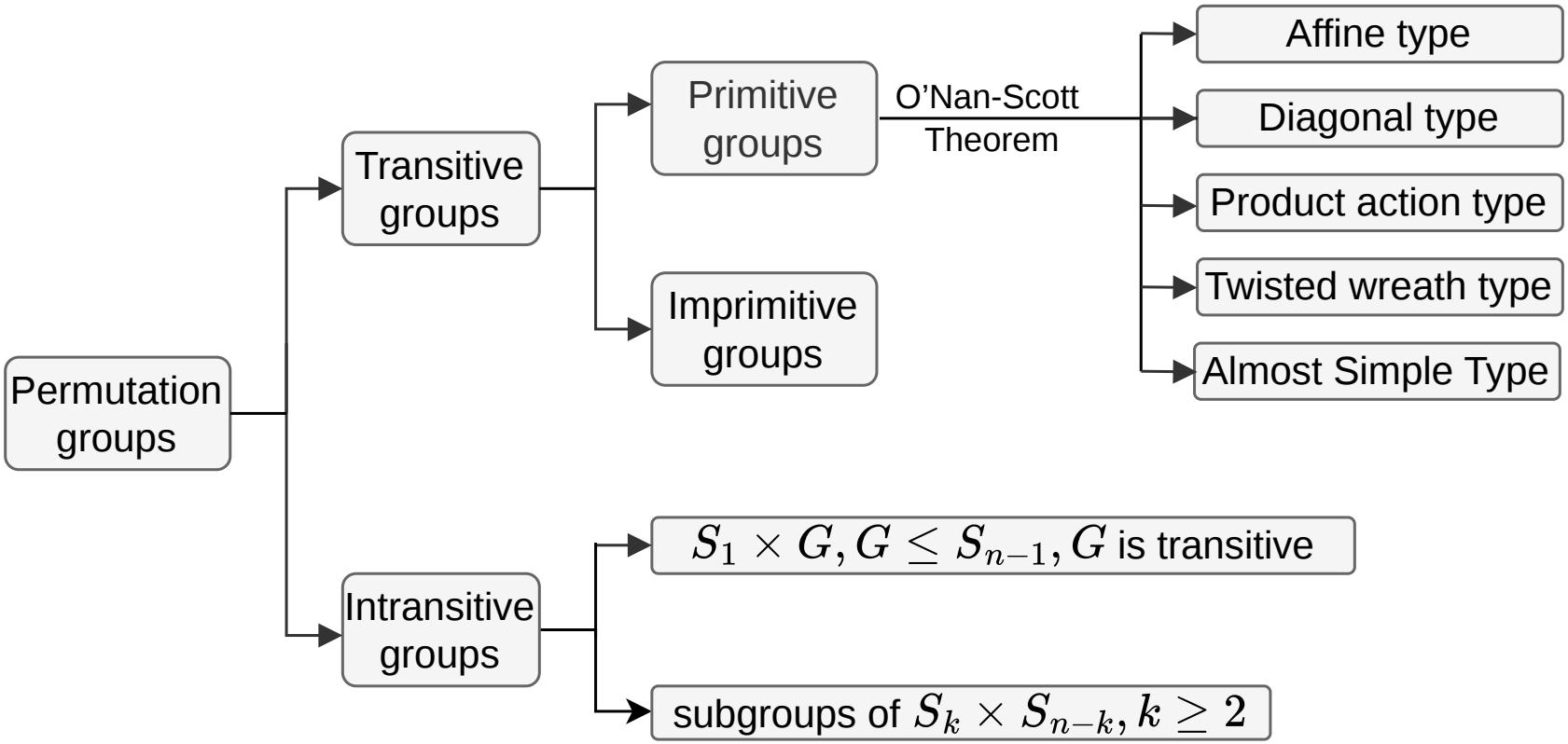
Orbit structure of $C_2 \mathrm{wr}_3 C_3$ =<(156243),(156)(243)>



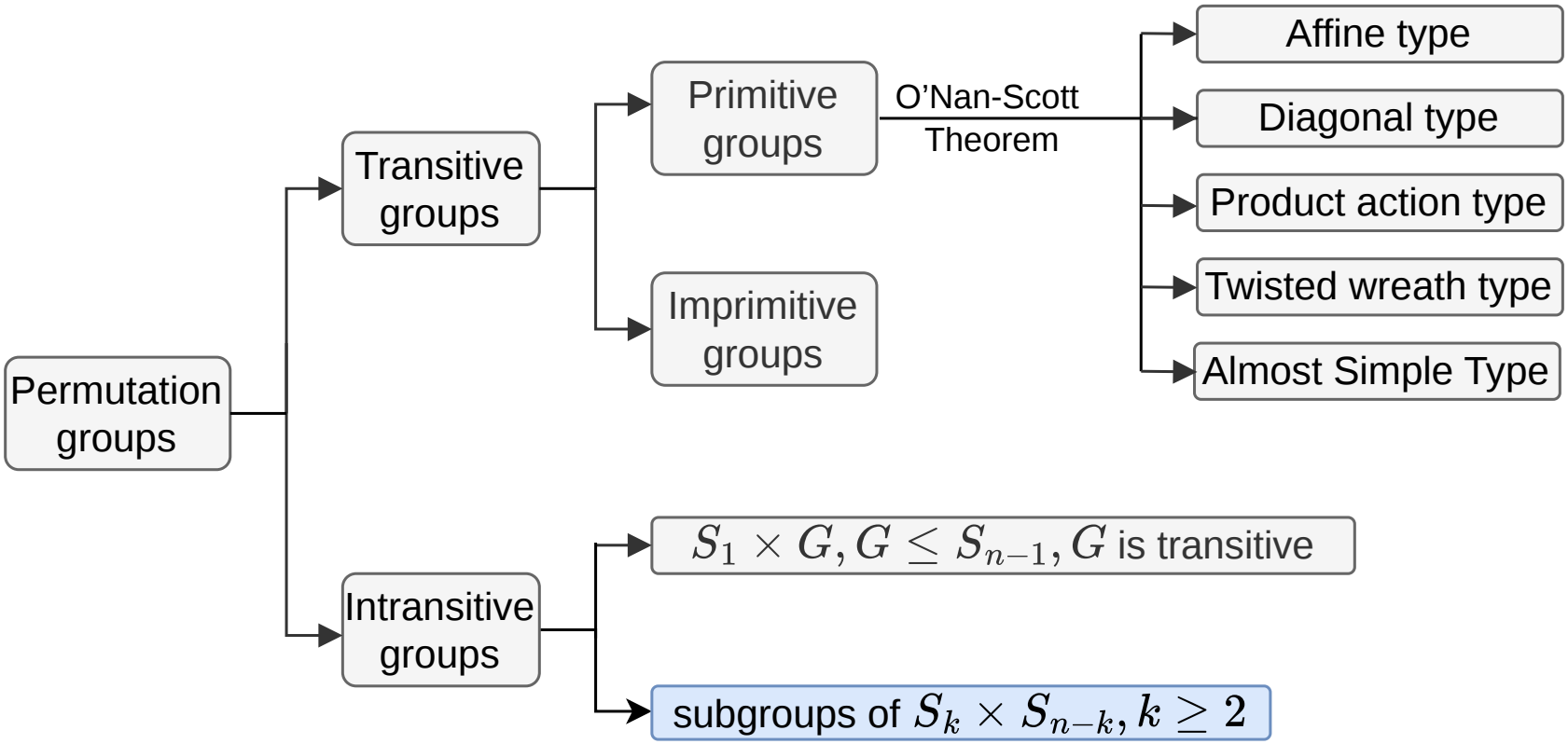
contract {1} and then restrict {3,4,5,6}

The family of 2-subsets of the minor a $(1, 4, 1)$-partition

Classification of all permutation groups



Subgroups of $S_k \times S_{n-k}$, $k \geq 2$

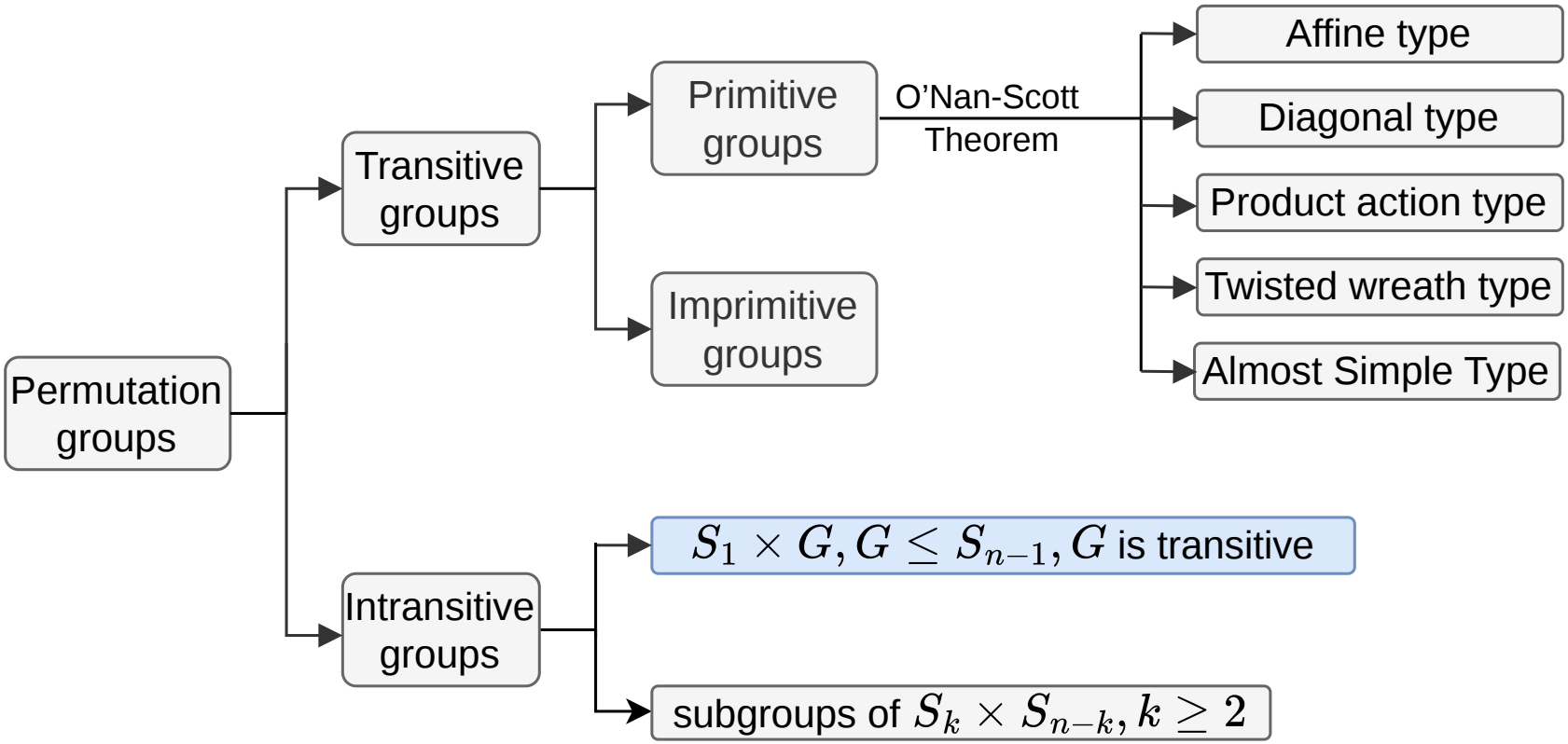


Ψ_G^* with $G \leq S_k \times S_{n-k}$, $k \geq 2$

Ψ_G^* is Shannon-not-tight

- Each G is a subgroup of some $S_k \times S_{n-k}$, whose restriction is $S_2 \times S_2$

$S_1 \times G$ with G transitive

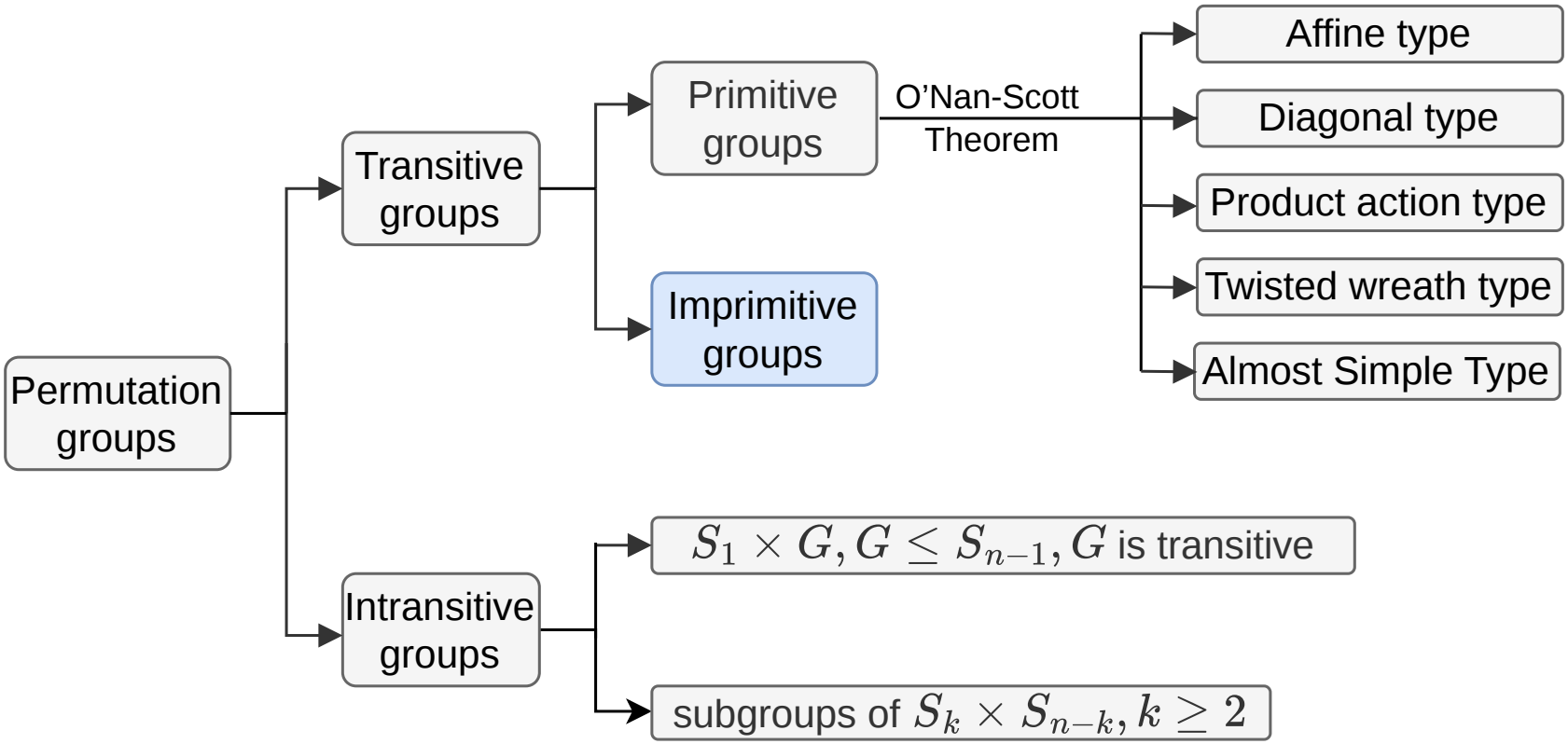


$\Psi^*_{S_1 \times G}$ with G transitive

$\Psi^*_{S_1 \times G}$ is

- 2-homogenous
  - Shannon-tight when $G = S_{n-1}$
  - Shannon-not-tight when $G = \mathrm{AGL}(1,7), \mathrm{PGL}(2,5), \mathrm{PGL}(2,8), \mathrm{PSL}(2,8)$
  - Unknow o.w.
- Shannon-not-tight when G is not 2-homogenous

Imprimitive G



Imprimitive G

Definition

A transitive permutation group G is imprimitive if there exists a partition $\{B_1, B_2, \ldots, B_t\}$ of N_n such that

- $|B_i| = k \geq 2$, $k|n$ and
- $\sigma B_i = B_j$ for each $\sigma \in G$.

Otherwise, it is called primitive.

Imprimitive G

Definition

A transitive permutation group G is imprimitive if there exists a partition $\{B_1, B_2, \ldots, B_t\}$ of N_n such that

- $|B_i| = k \geq 2$, $k|n$ and
- $\sigma B_i = B_j$ for each $\sigma \in G$.

Otherwise, it is called primitive.

Example

Cyclic group $C_4 = \langle (1234) \rangle$ is imprimitive as

- the partition $\{\{1, 3\}, \{2, 4\}\}$ is preserved.

Symmetric entropy regions induced by transitive group and intransitive groups

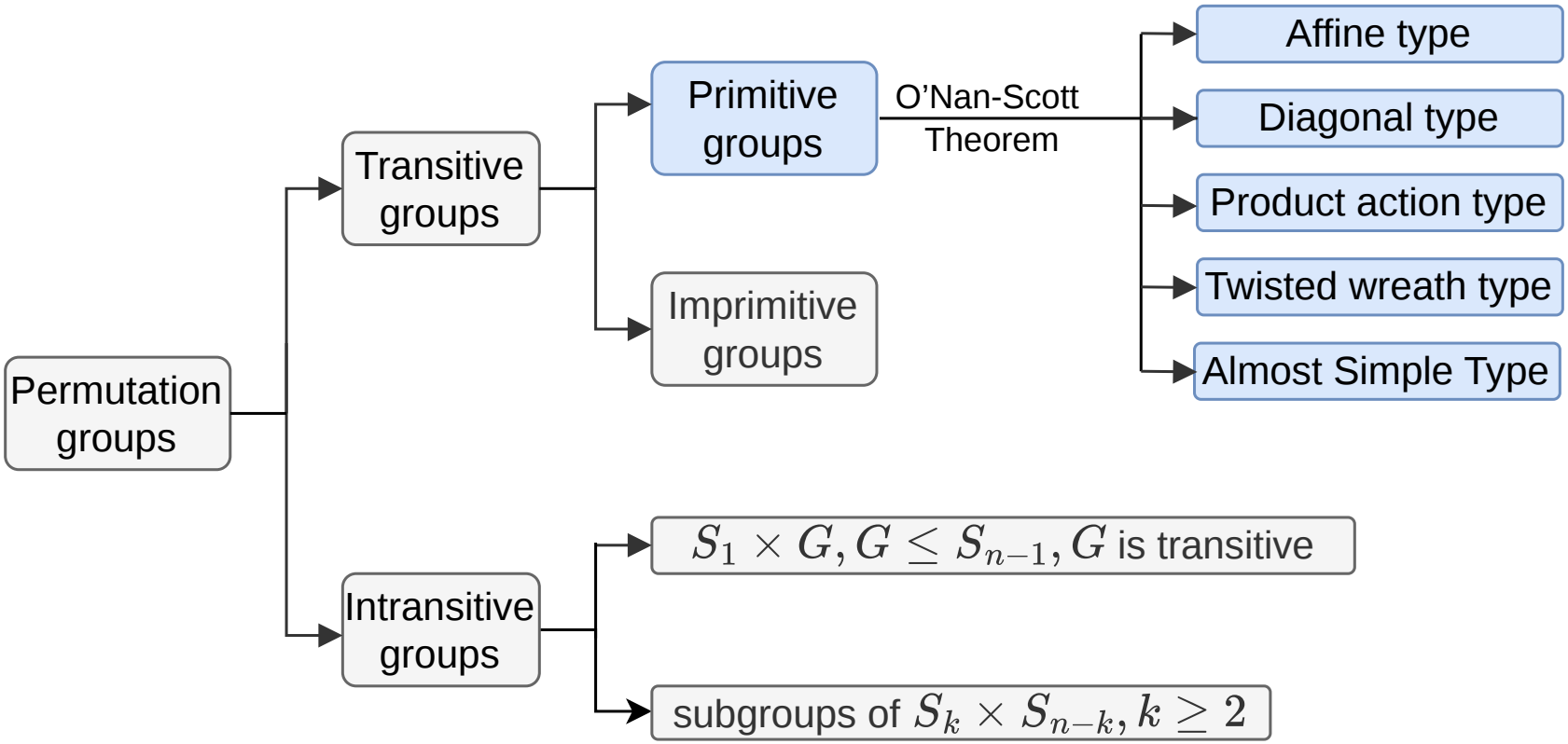
Theorem

If G is an imprimitive group acting on $\Omega = N_n$ with $n \geq 6$, then $\overline{\Psi^*_G} \neq \Psi_G$.

Proof sketch.

Each imprimitive G is a subgroup of $S_k \mathrm{wr} S_t$ with $n = kt, k, t \geq 2$, where $S_k \mathrm{wr} S_t$ the wreath product of S_k and S_t whose orbit structure has a minor finer than the rank structure of Vámos polymatroid. □

G is primitive



O'Nan Scott theorem: Classification of primitive groups

Theorem (O'Nan–Scott Theorem (Simplified version))

Let $G \leq S_n$ be primitive, then G is one of the following types

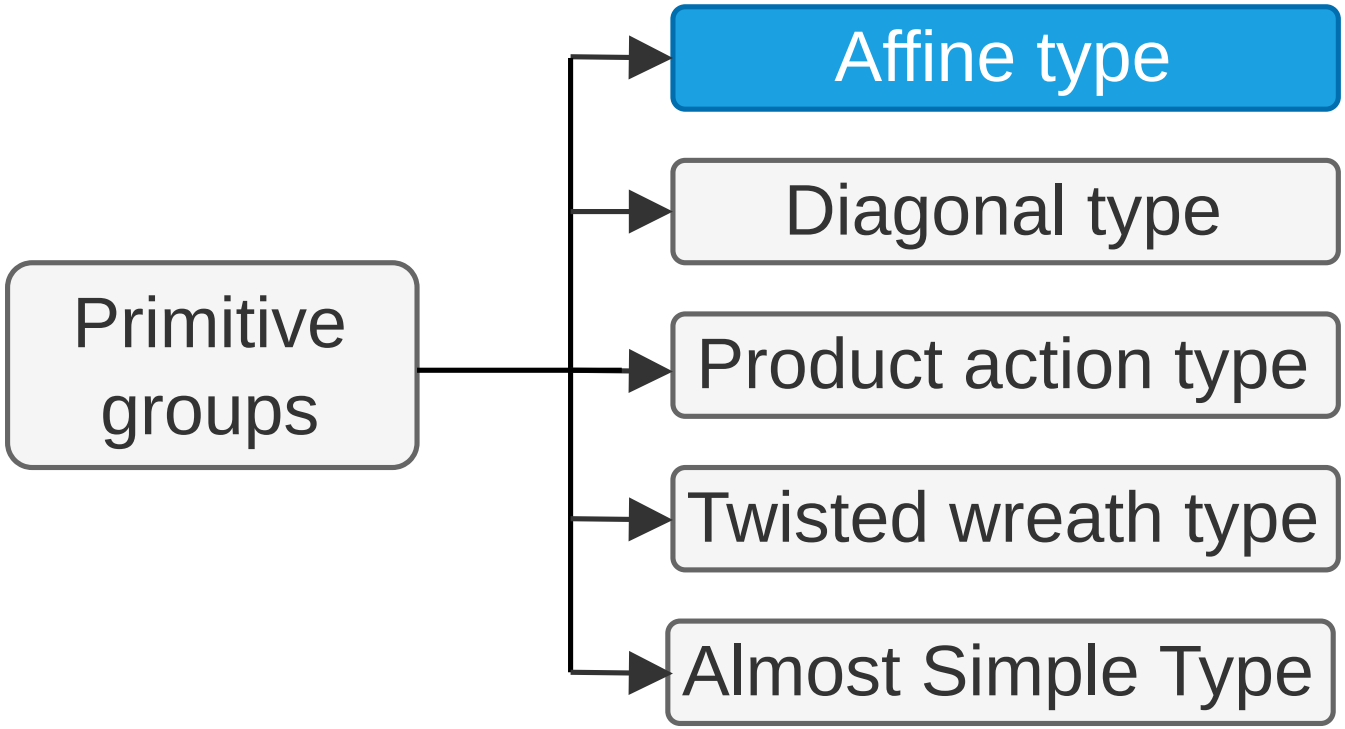
- *affine type*
- *diagonal type*
- *product action type*
- *twisted wreath type*
- *almost simple type*

according to its socle

$$\mathrm{soc}(G) = N_1 \times N_2 \times \cdots \times N_t$$

where N_i are minimal normal subgroups of G.

Affine type primitive groups



Affine type primitive groups

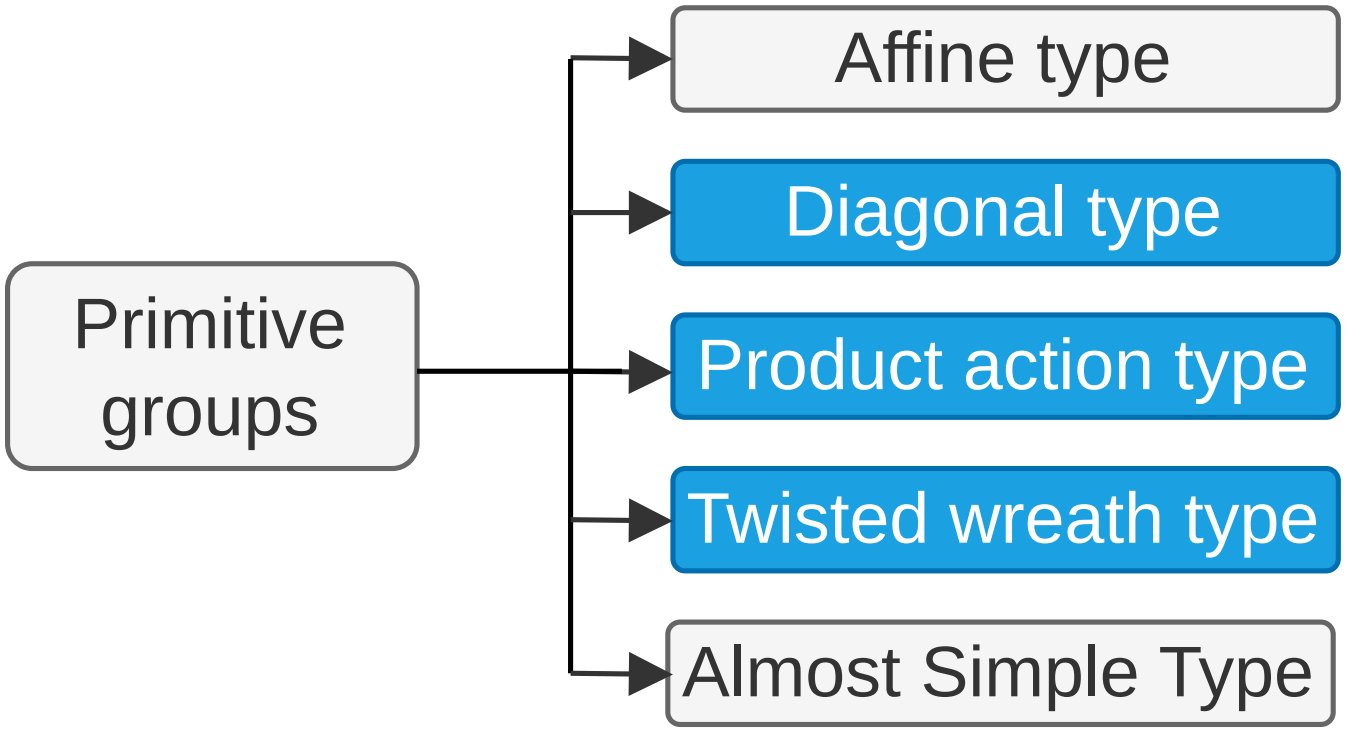
Group $G \leq \mathrm{AGL}(d, p)$ is called affine-type, where $\mathrm{AGL}(d, p)$ is the affine general linear group over $V = \mathbb{F}_p^d$. It's socle V is abelian.

Theorem

$\overline{\Psi^*_{\mathrm{AGL}(d,p)}} = \Psi_{\mathrm{AGL}(d,p)}$ *if and only if*

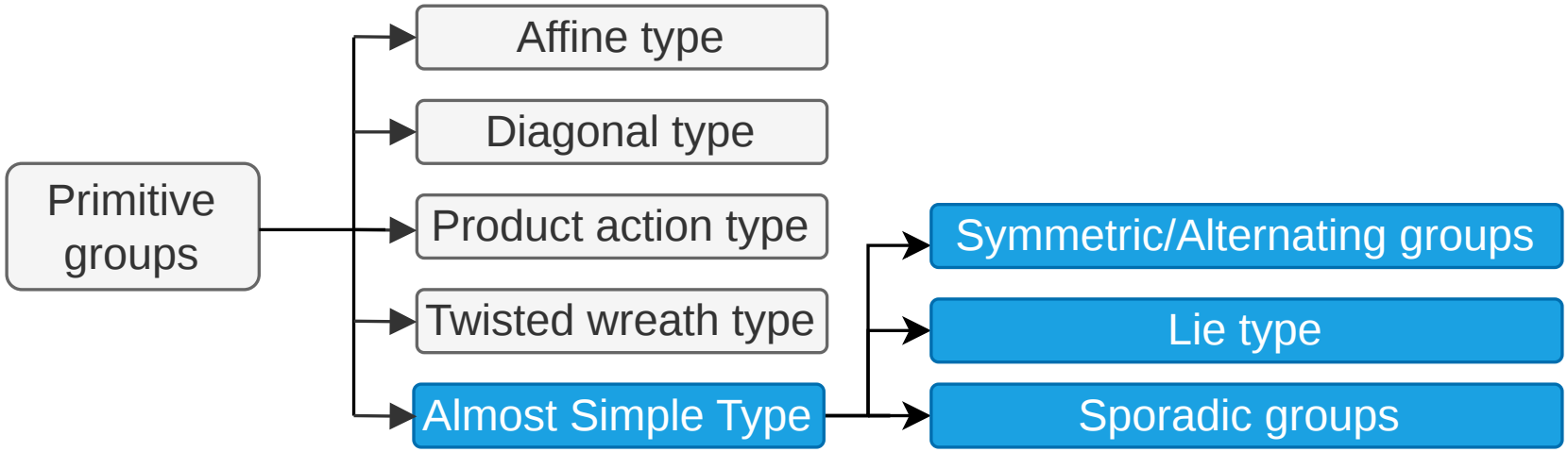
- $d = 1$ *and* $p \leq 7$
- $d = 2$ *and* $p = 2$.

Three types of primitive groups



For all groups G in these three primitive types, Ψ_G^* is Shannon-not-tight.

Almost simple type primitive groups



Almost simple type primitive groups

A primitive group $G \leq S_n$ is called almost simple if there exists a nonabelian simple group T such that

$$T \leq G \leq \mathrm{Aut}(A).$$

T is the socle of G.

Almost simple type primitive groups

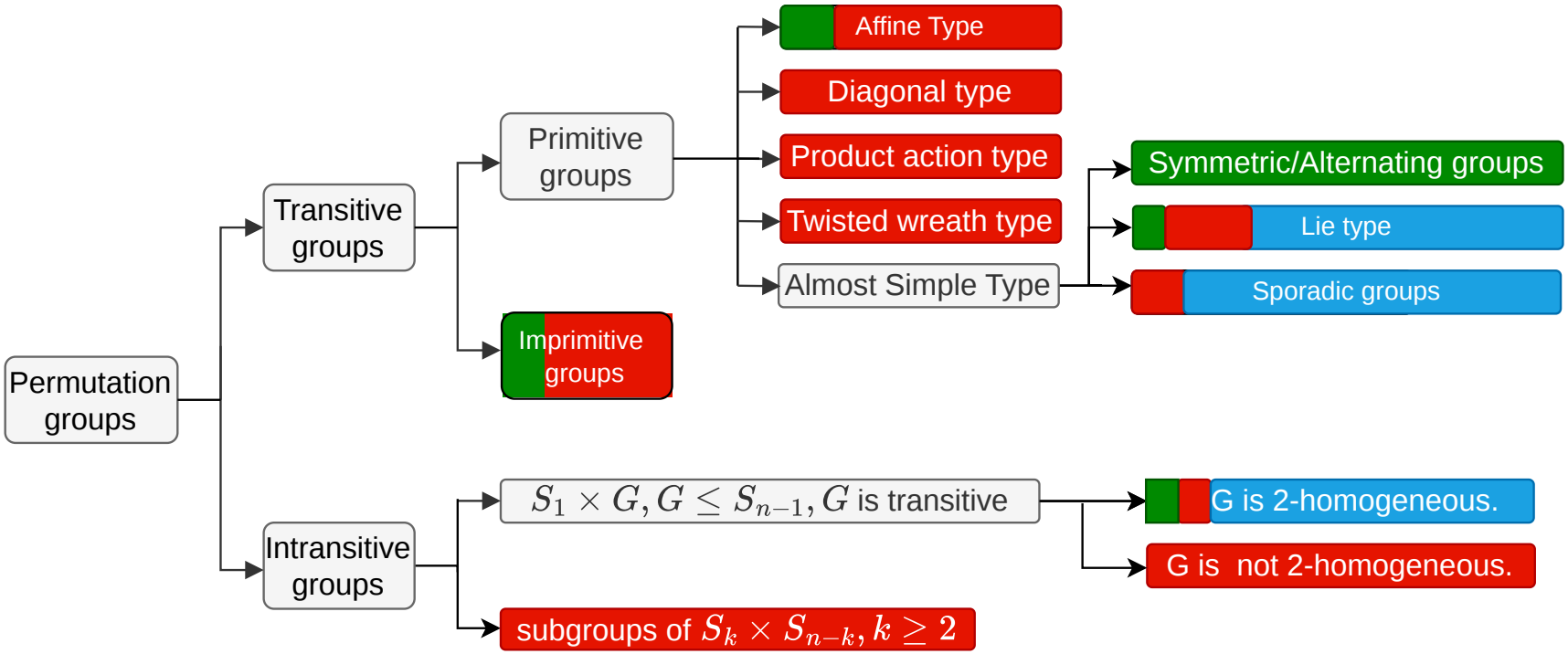
A primitive group $G \leq S_n$ is called almost simple if there exists a nonabelian simple group T such that

$$T \leq G \leq \mathrm{Aut}(A).$$

T is the socle of G.

- symmetric groups and alternating groups, Shannon-type
- Lie-type,
  - $\mathrm{PSL}(d, q)$ with $d = 2$, $q \leq 9, q \neq 7$ Shannon-tight, o.w. Shannon-not-tight
  - $\mathrm{PGL}, \mathrm{PSU}, \ldots$ unknown
- Sporadic groups
  - Mathieu groups, J_1, J_2, HS, Shannon-not-tight
  - others, unknown

Conclusion


Permutation groups
Transitive groups
Intransitive groups
Primitive groups
Imprimitive groups
Affine Type
Diagonal type
Product action type
Twisted wreath type
Almost Simple Type
Symmetric/Alternating groups
Lie type
Sporadic groups
$S_1 \times G, G \leq S_{n-1}, G$ is transitive
subgroups of $S_k \times S_{n-k}, k \geq 2$
G is 2-homogeneous.
G is not 2-homogeneous.

Two conjectures

Conjecture

For a Shannon-tight Ψ^*_G, its closure $\overline{\Psi^*_G}$ is multi-linear.

Two conjectures

Conjecture

For a Shannon-tight Ψ^*_G, its closure $\overline{\Psi^*_G}$ is multi-linear.

Conjecture

The converse of forbidden minor lemma is also true: For any $G \leq S_n$, $\overline{\Psi^*_G} \neq \Psi_G$ if and only if $\mathcal{O}_F$ has a minor on a 4-subset $\Omega \subseteq N_n$ which is a refinement of the rank structure of Vámos polymatroid.

Acknowledgement

- Nigel Boston, who told me O'Nan Scott theorem which made the general case possible
- ChatGPT, who
  - teaches me O'Nan Scott theorem and theory of permutation groups, and
  - helps us do the proofs of some cases.

Thank you!

Questions?