Space Information Flow

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(joint work with Chuan Wu @ HKU)

Talk Outline

1. Space Information Flow in a NutShell
2. Space Information Flow: Motivation
3. Multiple Unicast in Space
4. Multicast in Space
5. Conclusion and Open Problems

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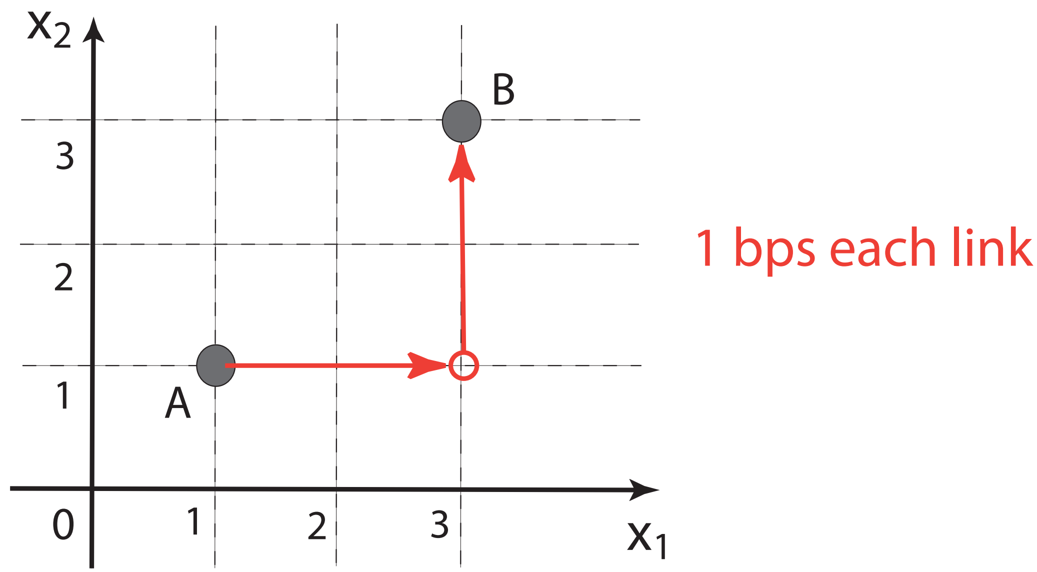
1-1. Space Information Flow: The Problem

Space Information Flow:

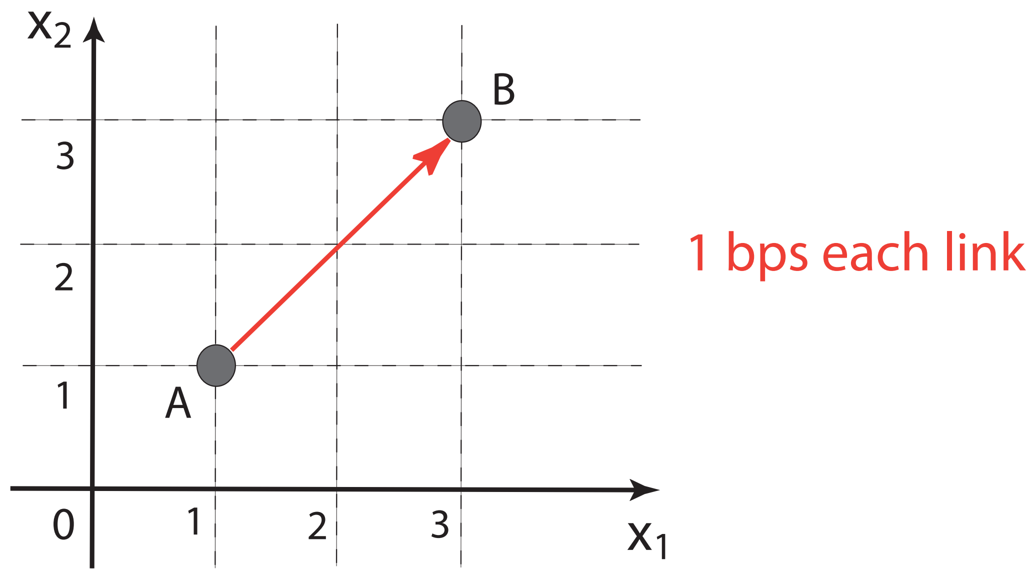
- Transmit information flows in space to satisfy end-to-end (unicast/multicast) communication demands among terminals at known coordinates
- Minimize $\sum_e (\mathbf{f}_e ||e||)$
  - e: a 'link' employed by the flow $\mathbf{f}$
  - $\mathbf{f}_e$: flow rate at e
  - $||e||$: length of e

1-2. Space Information Flow: Unicast Example



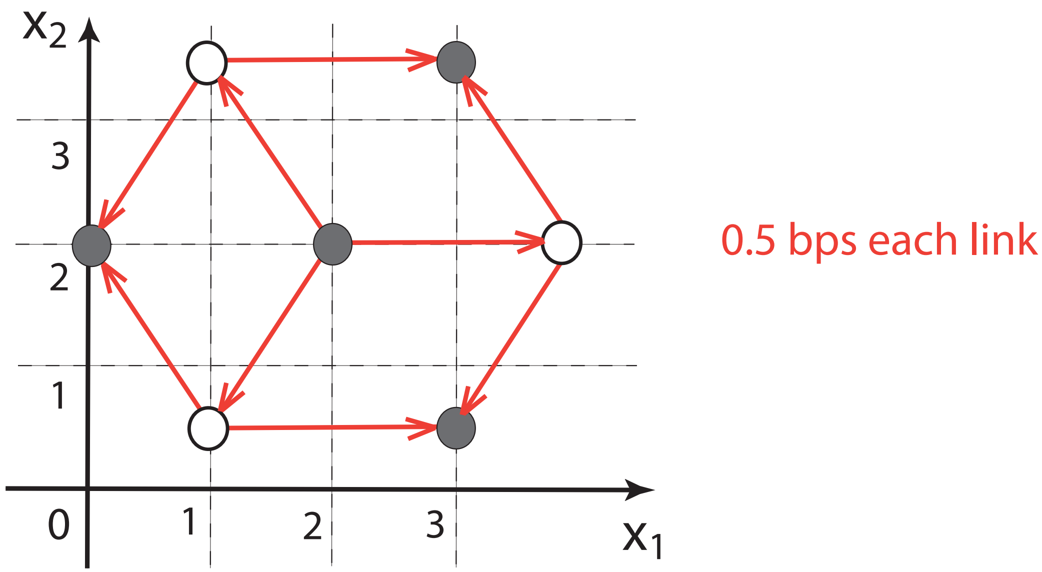
- Unicast demand: $A \rightarrow B$, 1 bps
- Cost: $(2m + 2m) \times 1\, bit/sec = 4\ bit \cdot meter/sec$

1-3. Space Information Flow: Unicast Example



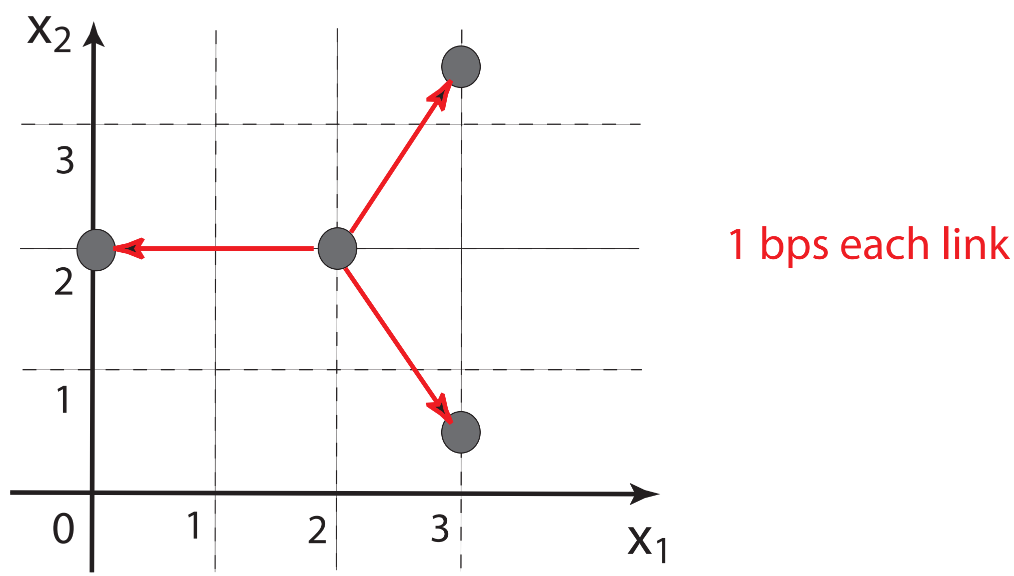
- Unicast demand: $A \rightarrow B$, 1 bps
- Cost: $(2\sqrt{2}\,m) \times (1\,bit/sec) = 2.828\ bit \cdot meter/sec$

1-4. Space Information Flow: Multicast Example



- Multicast demand among terminal (black) nodes, 1 bps
- Cost: $(2\,m) \times (0.5\,bit/sec) \times 9 = 9\;bit \cdot meter/sec$

1-5. Space Information Flow: Multicast Example



- Multicast demand among terminal (black) nodes, 1 bps
- Cost: $(2\,m) \times (1\,bit/sec) \times 3 = 6\ bit \cdot meter/sec$

1-6. The (Geometric) Steiner Tree Problem

Gauss, 1836: *how can a railway network of minimal length which connects the four German cities Bremen, Hamburg, Hannover, and Braunschweig be created?*

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1.7. Space Information Flow vs. Steiner Tree

- SIF allows fractional flow rates
  - Steiner tree (implicit): each link has flow rate 1.0
- SIF allows information encoding (network coding)
  - Steiner tree: can model replication, but no coding
- SIF allows multiple sessions (with inter-session coding)

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2-1. Network Coding *vs.* Routing

The space model represents the 'fairest' paradigm for comparing network coding and routing.

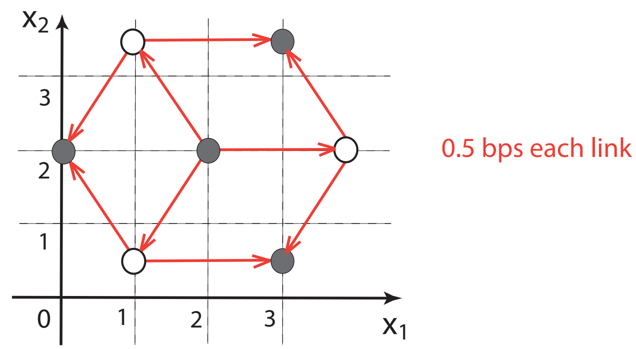
- Directed networks: contrived, tailored network topology and orientation favoring network coding
- Undirected networks: contrived, tailored network topology favoring network coding
- Space: network coding and routing are each free to design its own network topology and choose its own network orientation

2-2. Network Coding *vs.* Routing

The space model represents the 'fairest' paradigm for comparing network coding and routing.

	directed networks	undirected networks	space
multiple unicast	∞ $\Omega(n)$	conjectured: $\equiv 1$	$\equiv 1$
multicast	∞ $\Omega(\sqrt{n})$	≤ 2 $\geq 8/7$	≤ 1.155 (2-D) $\geq \alpha \in (1, 1.022)$

2-3. Information Network Design



- A space information flow **f** can be viewed as a blue print for constructing an information network
- A link e in **f** — a communication cable to be laid
- Link flow rate $\mathbf{f}_e$ — bandwidth capacity of the cable
- The longer the cable, the more expensive
- The wider the cable, the more expensive

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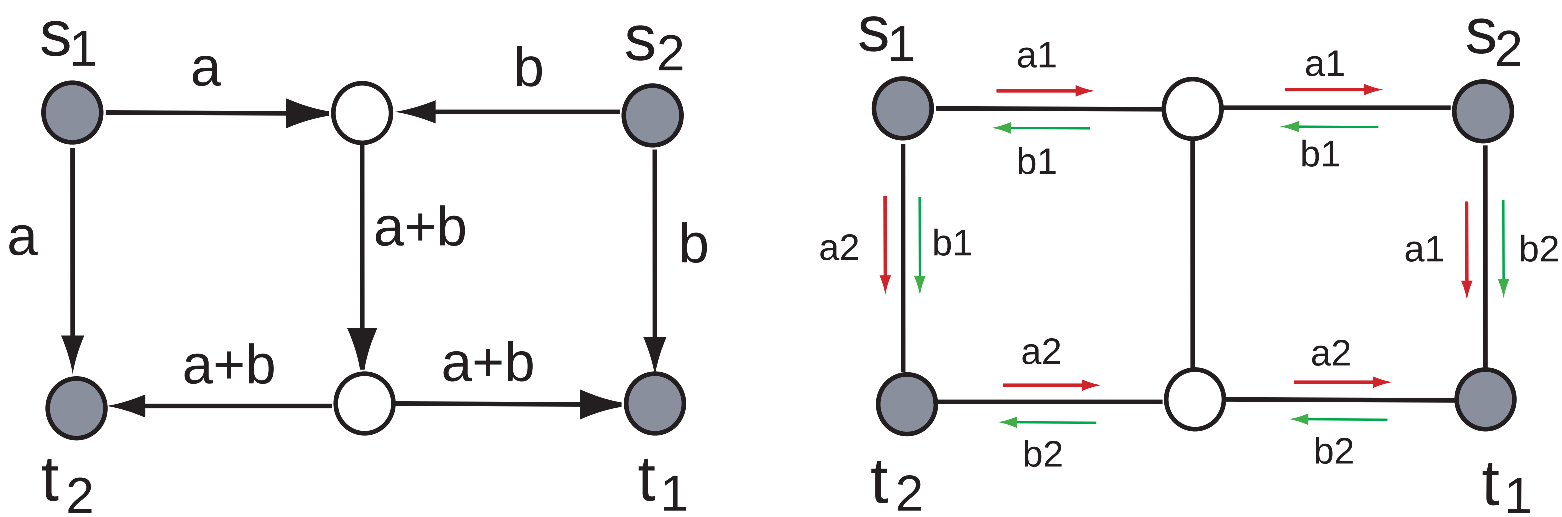
3-1. Network Information Flow: Two Unicast

(A) each link flow rate is 1.0

(B) each link flow rate is 0.5



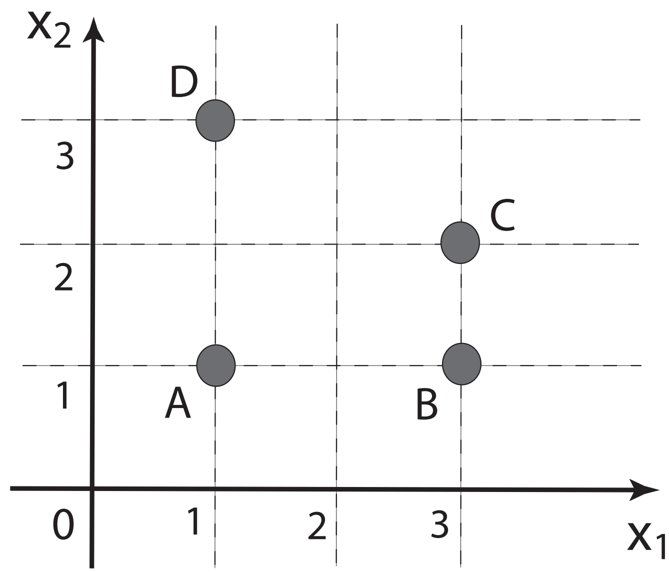
3-2. The Multiple Unicast Conjecture

Throughput domain: *For k independent unicast sessions in a capacitated undirected network $(G, \mathbf{c})$, a throughput vector $\mathbf{r}$ is feasible with network coding if and only if it is feasible with routing.*

$$\Updownarrow$$

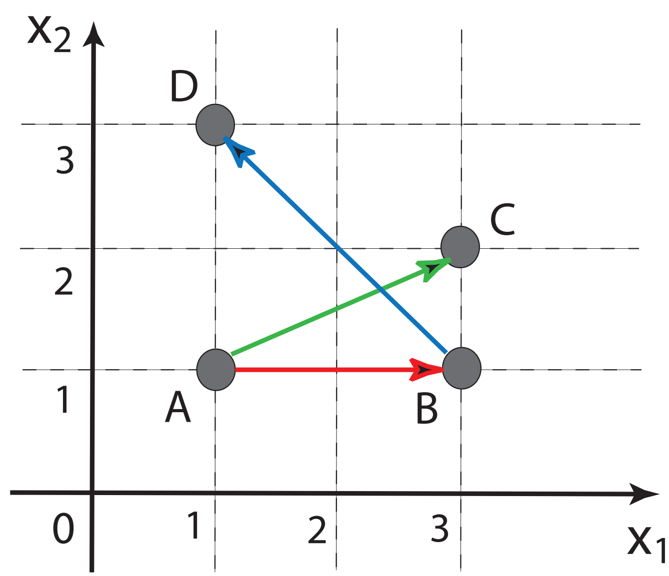
Cost domain: *Let $\mathbf{f}$ be the underlying flow vector of a network coding solution for k independent unicast sessions with throughput vector $\mathbf{r}$, in a cost-weighted undirected network $(G, \mathbf{w})$. Then $\sum_e \mathbf{w}_e \mathbf{f}_e \geq \sum_i \mathbf{d}_i \mathbf{r}_i$, where $\mathbf{d}_i$ is the shortest path distance between the sender and receiver of session i under metric $\mathbf{w}$.*

3-3. Multiple Unicast in Space: The Problem



What's the best solution for three unicast: $A \to B$, $A \to C$, $B \to D$, each with unit throughput demand?

3-4. Multiple Unicast in Space: The Problem



$$Cost = \sum_i \mathbf{r}_i \mathbf{d}_i$$

Is optimal cost without network coding still optimal with network coding?

3-5. Multiple Unicast in Space: The Theorem

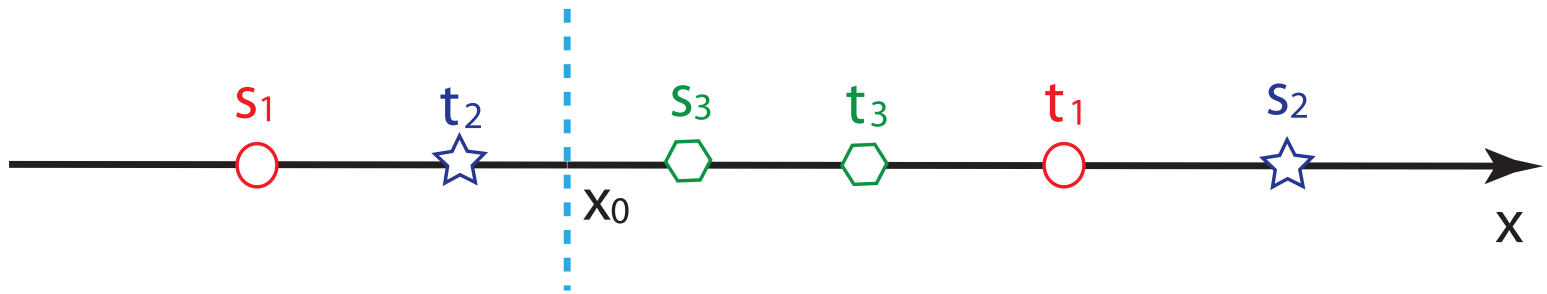
Theorem. For multiple unicast in space, network coding is equivalent to routing.

We prove the cost version of the multiple unicast conjecture in space.

$$Prove : \sum_e \mathbf{f}_e ||e|| \geq \sum_i \mathbf{r}_i \mathbf{d}_i$$

$\mathbf{f}$: the underlying flow vector of a network coding solution.

3-6. Multiple Unicast in 1-D Space

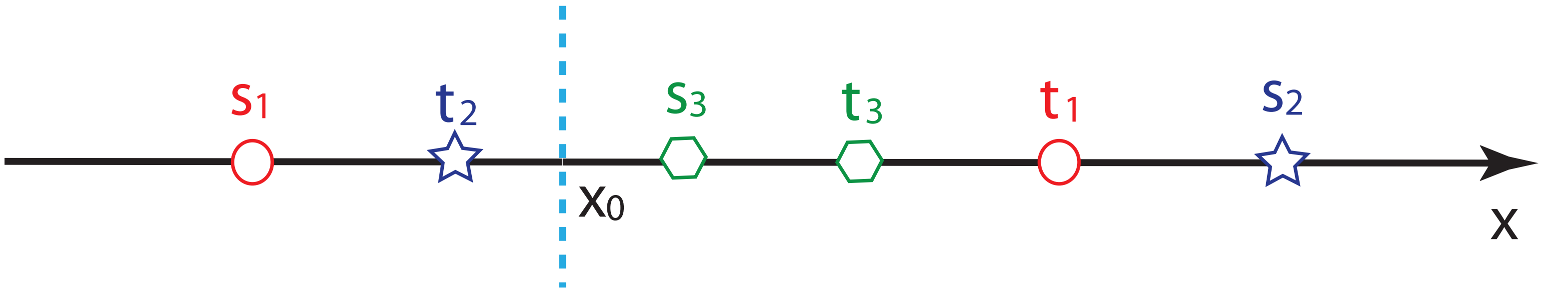


Throughput demand:

- $s_1 \to t_1$: $\mathbf{r}_1$
- $s_2 \to t_2$: $\mathbf{r}_2$
- $s_3 \to t_1$: $\mathbf{r}_3$

To prove: $\sum_e (||e||_1 \mathbf{f}_e) \geq \sum_i (||s_i t_i||_1 \mathbf{r}_i)$.

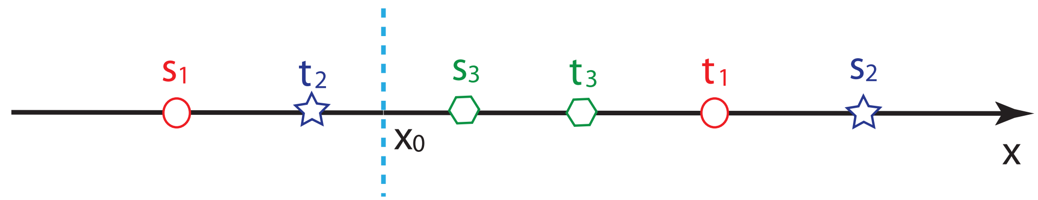
3-7. Multiple Unicast in 1-D Space



$$\mathbf{f}_{x_0} \geq Demand((-\infty, x_0) \leftrightarrow (x_0, \infty))$$

$$= \mathbf{r_1} + \mathbf{r_2}$$

3-8. Multiple Unicast in 1-D Space



$$\int_{x=-\infty}^{\infty} \mathbf{f}_x dx \geq \int_{x=-\infty}^{\infty} Demand((-\infty, x) \leftrightarrow (x, \infty))dx$$

$$LFH = \sum_e (||e||_1 \mathbf{f}_e)$$

$$RHS = \sum_i ||s_i t_i||_1 \mathbf{r}_i$$

Therefore: $\sum_e (||e||_1 \mathbf{f}_e) \geq \sum_i (||s_i t_i||_1 \mathbf{r}_i)$.

3-9. Multiple Unicast in h-D Space

To prove: $\sum_e (\mathbf{f}_e ||e||_h) \geq \sum_i (||s_i t_i||_h \mathbf{r_i})$

Assume, b.w.o.c.: $\sum_e (\mathbf{f}_e ||e||_h) < \sum_i (||s_i t_i||_h \mathbf{r_i})$

Find a unit 1-D vector $\vec{p}$, $s.t.$:

$$\sum_e (\mathbf{f}_e | e \cdot \vec{p} |) < \sum_i (| \overrightarrow{s_i t_i} \cdot \vec{p} | \mathbf{r_i})$$

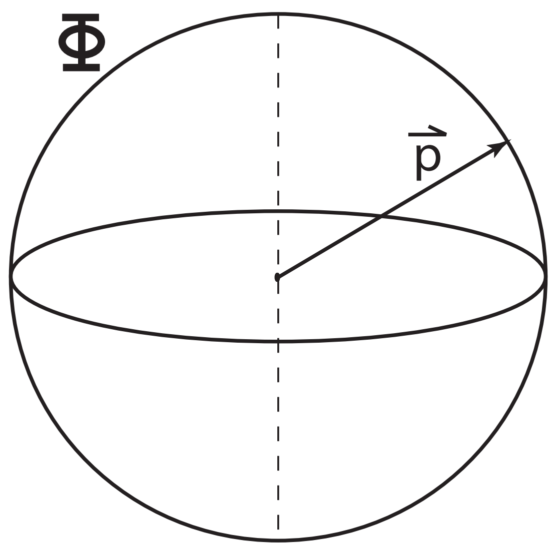
Contradiction with result in 1-D.

3-10: Multiple Unicast in h-D Space

Challenge: $\vec{p}$ is hard to find!

Idea: enumerate all possible $\vec{p}$, by integrating over Φ.
Prove:

$$\oiint_{\Phi} \sum_{e} (\mathbf{f}_e |e \cdot \vec{p}|) d\Phi < \oiint_{\Phi} \sum_{i} (|\vec{s_i t_i} \cdot \vec{p}| \mathbf{r}_i) d\Phi$$



3-11. Multiple Unicast in h-D Space

$$\oiint_{\Phi} \sum_{e} (\mathbf{f}_e |e \cdot \vec{p}|) d\Phi =_1 \sum_{e} \oiint_{\Phi} \mathbf{f}_e |e \cdot \vec{p}| d\Phi$$

$$=_2 \sum_{e} \oiint_{\Phi} \mathbf{f}_e ||e||_h | \vec{1} \cdot \vec{p}| d\Phi$$

$$=_3 \sum_{e} (\mathbf{f}_e ||e||_h) \oiint_{\Phi} | \vec{1} \cdot \vec{p}| d\Phi$$

3-12. Multiple Unicast in h-D Space

$$\oiint_\Phi \sum_i (\overrightarrow{s_i t_i} \cdot \overrightarrow{p}) \mathbf{r}_i d\Phi = \sum_i \oiint_\Phi |\overrightarrow{s_i t_i} \cdot \overrightarrow{p}| d\Phi$$

$$= \sum_e \oiint (||s_i t_i||_h |\overrightarrow{1} \cdot \overrightarrow{p}|) \mathbf{r}_i d\Phi = \sum_i (||s_i t_i||_h \mathbf{r}_i) \oiint_\Phi |\overrightarrow{1} \cdot \overrightarrow{p}| d\Phi$$

By assumption: $\sum_e (\mathbf{f}_e ||e||_h) < \sum_i (||s_i t_i||_h \mathbf{r}_i)$

We claim:

$$\oiint_\Phi \sum_e (\mathbf{f}_e |e \cdot \overrightarrow{p}|) d\Phi < \oiint_\Phi \sum_i (|\overrightarrow{s_i t_i} \cdot \overrightarrow{p}| \mathbf{r}_i) d\Phi$$

3-13. Multiple Unicast in h-D Space

We claim:

$$\oiint_{\Phi} \sum_{e} (\mathbf{f}_e |e \cdot \overrightarrow{p}|) d\Phi < \oiint_{\Phi} \sum_{i} (|\overrightarrow{s_i t_i} \cdot \overrightarrow{p}| \mathbf{r}_i) d\Phi$$

There must exist at least one particular $\overrightarrow{p}$, such that:

$$\sum_{e} (\mathbf{f}_e |e \cdot \overrightarrow{p}|) < \sum_{i} (|\overrightarrow{s_i t_i} \cdot \overrightarrow{p}| \mathbf{r}_i)$$

3-14. Multiple Unicast: Network *vs.* Space

- Isometric (distance-perserving) embedding of graph metric?
- Low-distortion embedding of graph metric?
- Using a Euclidean or non-Euclidean geometry

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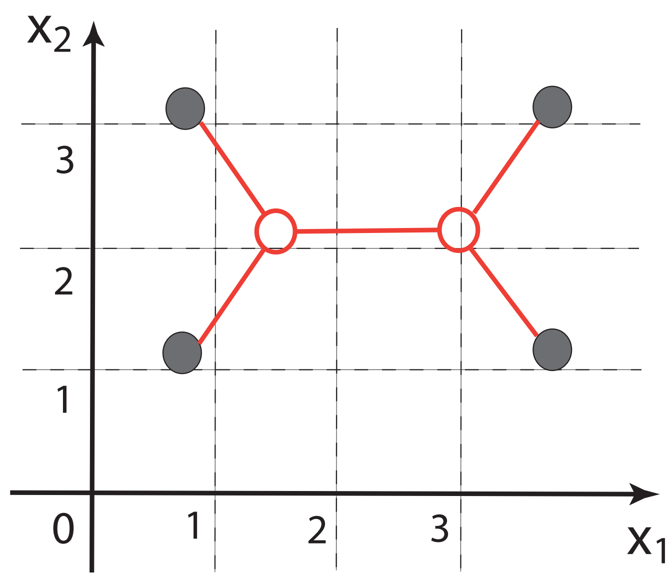
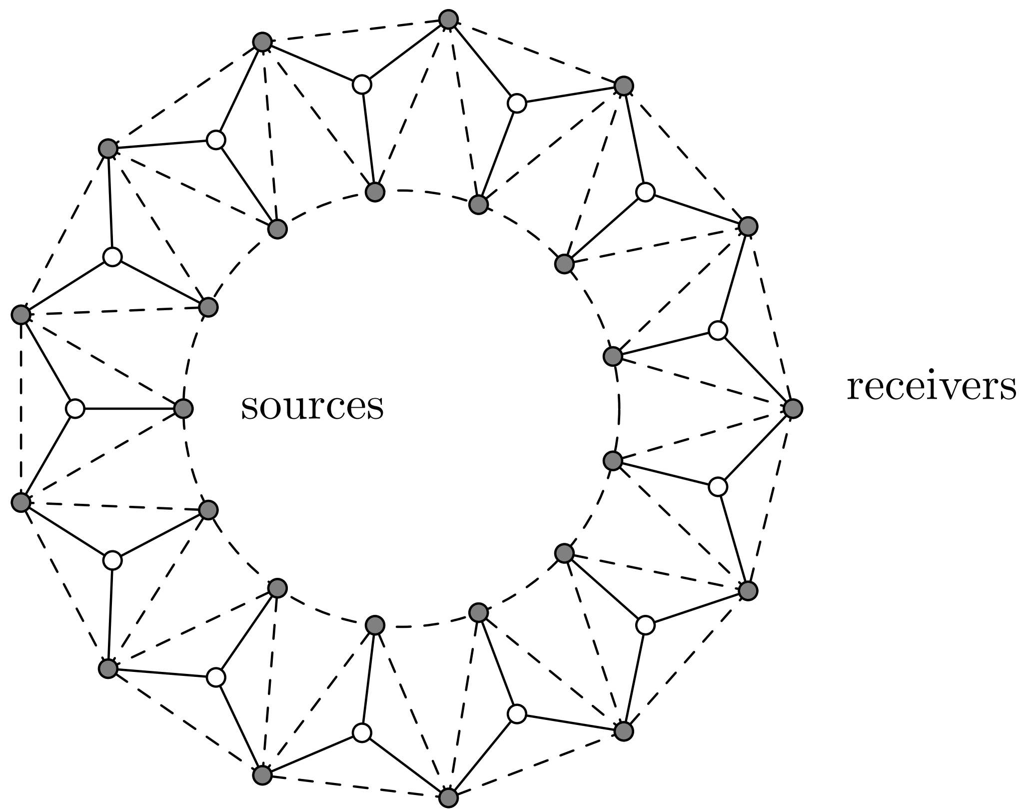
4-1. Multicast in Space

Is an optimal multicast solution in space always a multicast tree ?



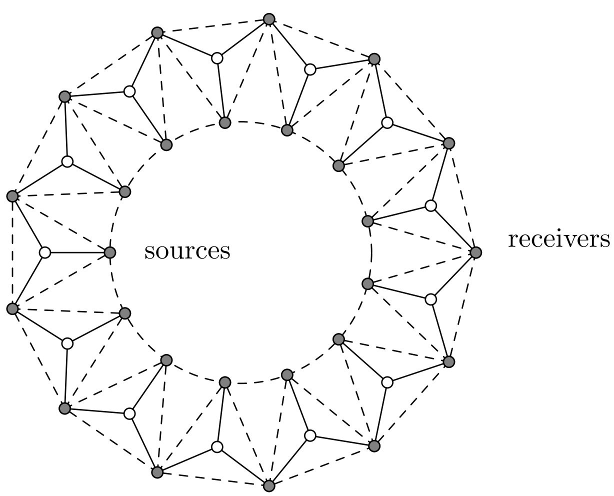
4-2. Multicast in Space

Is an optimal multicast solution in space always a multicast tree? — No!



Open problems:

- What is the computational complexity of the optimal multicast problem in space? P? NP-hard?
- Is this pattern minimum?



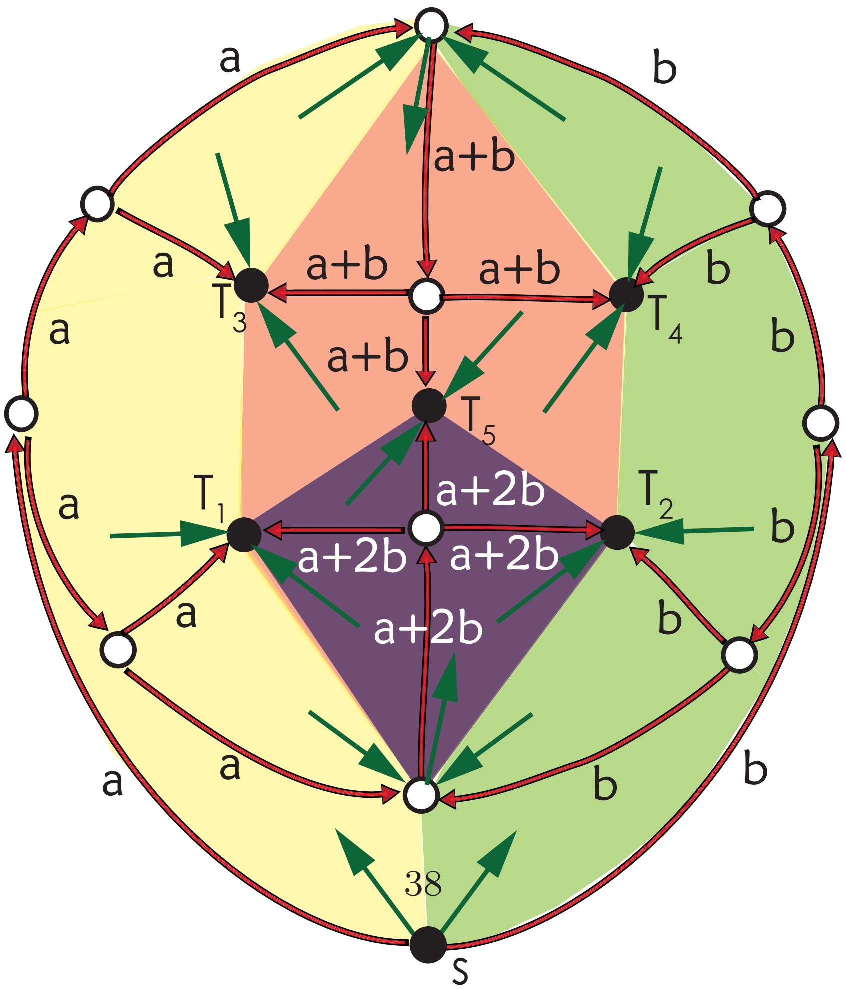
4-4. Multicasting 2 Flows in Space

Theorem. Multicasting 2 flows: cost advantage of network coding $\leq$ the Steiner ratio

- cost advantage: the ratio of: min multicast tree cost over min multicast cost with network coding
- Steiner ratio: the ratio of: min spanning tree cost over min Steiner tree cost
- The Gilbert-Pollak Conjecture: the Steiner ratio in 2-D is at most $\frac{2}{\sqrt{3}} = 1.155$.

4-5. Multicasting 2 Flows in Space

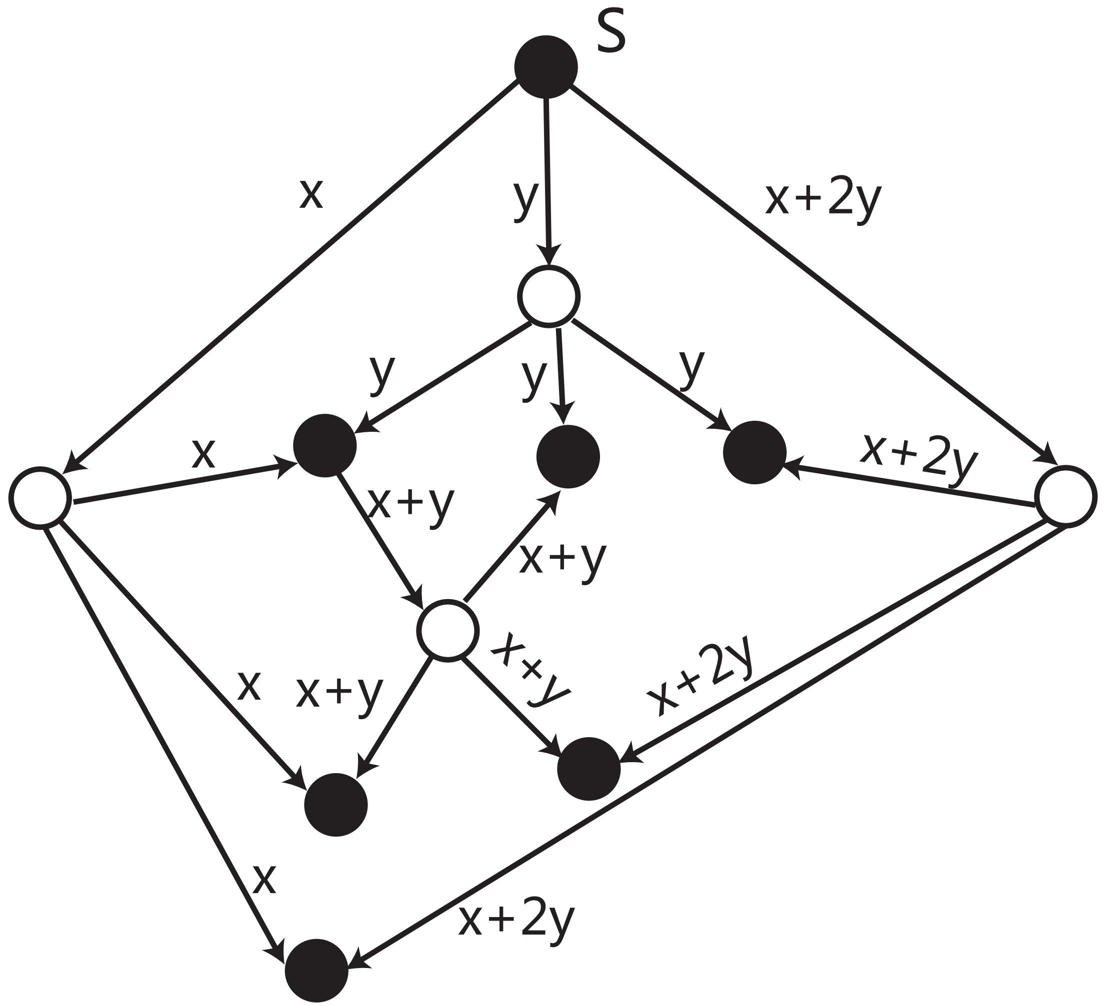
- Multicast flow decomposition
- Replace each component with a local spanning tree
- Resulting network is essentially a broadcast network



4-6. Bipartite Multicast Flow Structure

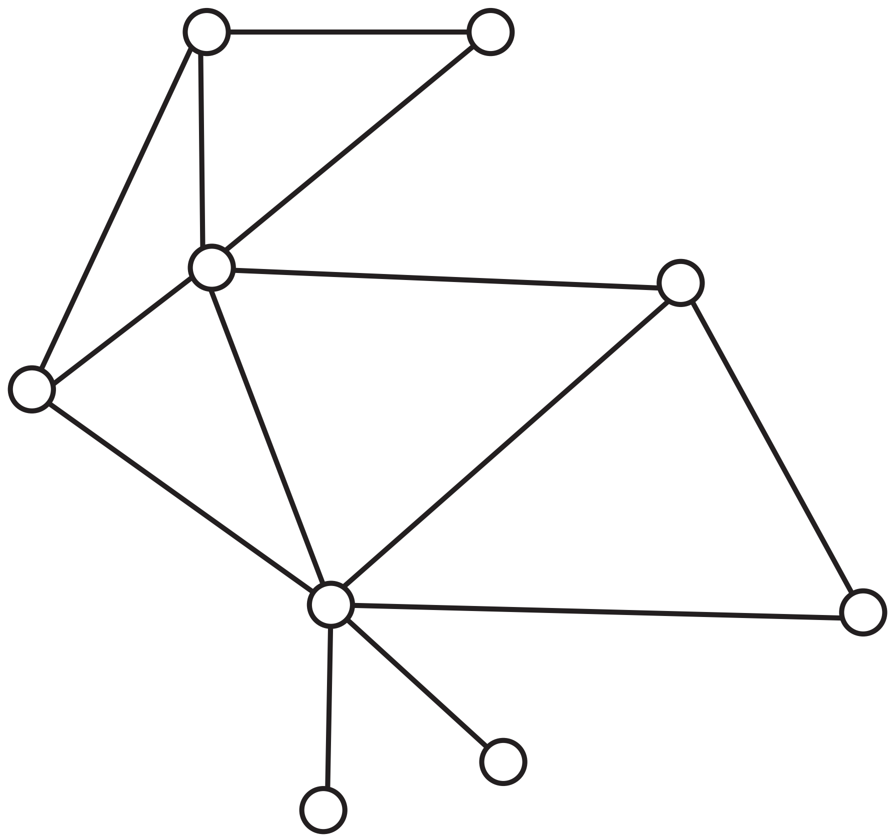
Theorem. Network coding solution has a bipartite structure: cost advantage of network coding $\leq$ 1.155 (unconditional).

4-7. Bipartite Multicast Flow Structure

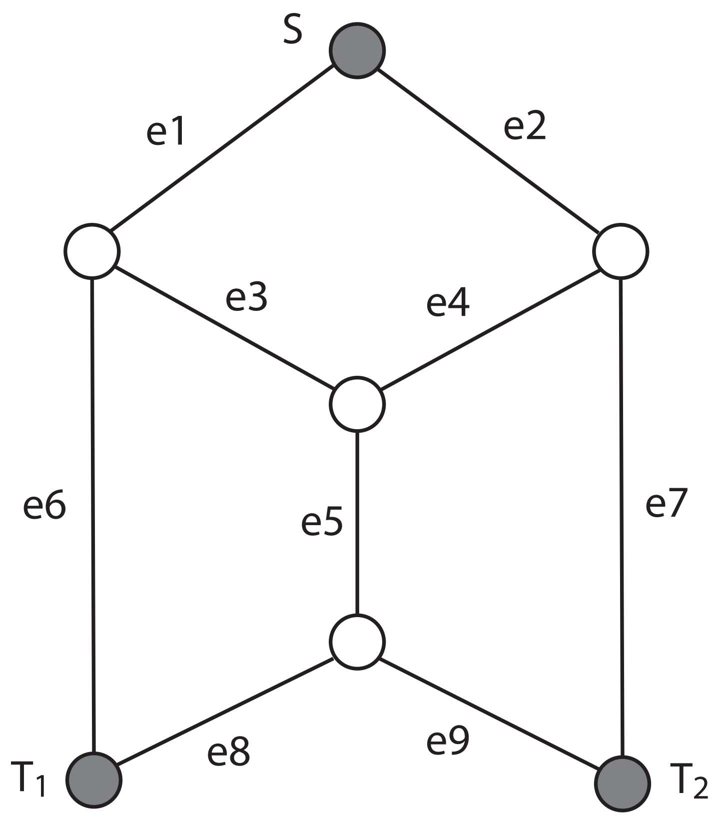


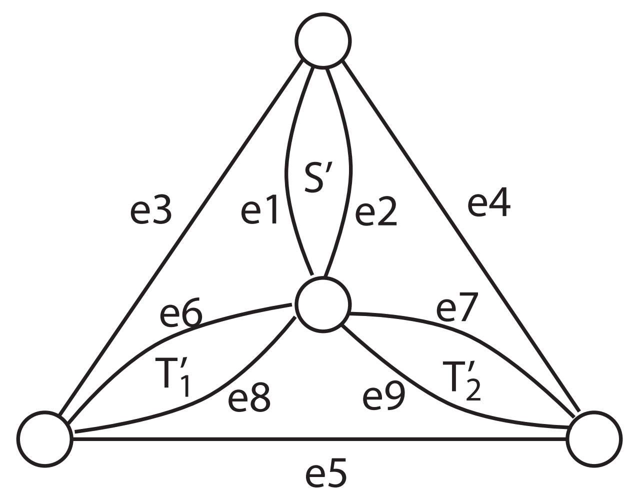
Outer-planar (all nodes on same face):

Network Coding = Routing



Terminals on Same Face: $GF(2)$ Sufficient?





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5-1. Conclusion

- Introduced the space information flow problem
- SIF models information network design
- Proved the multiple unicast conjecture in space
- Proved upper-bounds for benefits of multicast network coding in space

5-2. Open Problems for Space Information Flow

1. Optimal multicast in space: complexity, efficient algorithm design

2. Extend upper-bound analysis for multicast coding advantage in space

3. Wireless information flow in space, wireless network design

4. Multiple unicast, embedding?

5. Multicast in planar networks: small fields suffice?

5-2. Open Problems for Space Information Flow

1. Optimal multicast in space: complexity, efficient algorithm design
2. Extend upper-bound analysis for multicast coding advantage in space
3. Wireless information flow in space, wireless network design
4. Multiple unicast, embedding?
5. Multicast in planar networks: small fields suffice?

THE END