Network Coding in Planar Networks

Zongpeng Li

April 18, 2012

joint work with:

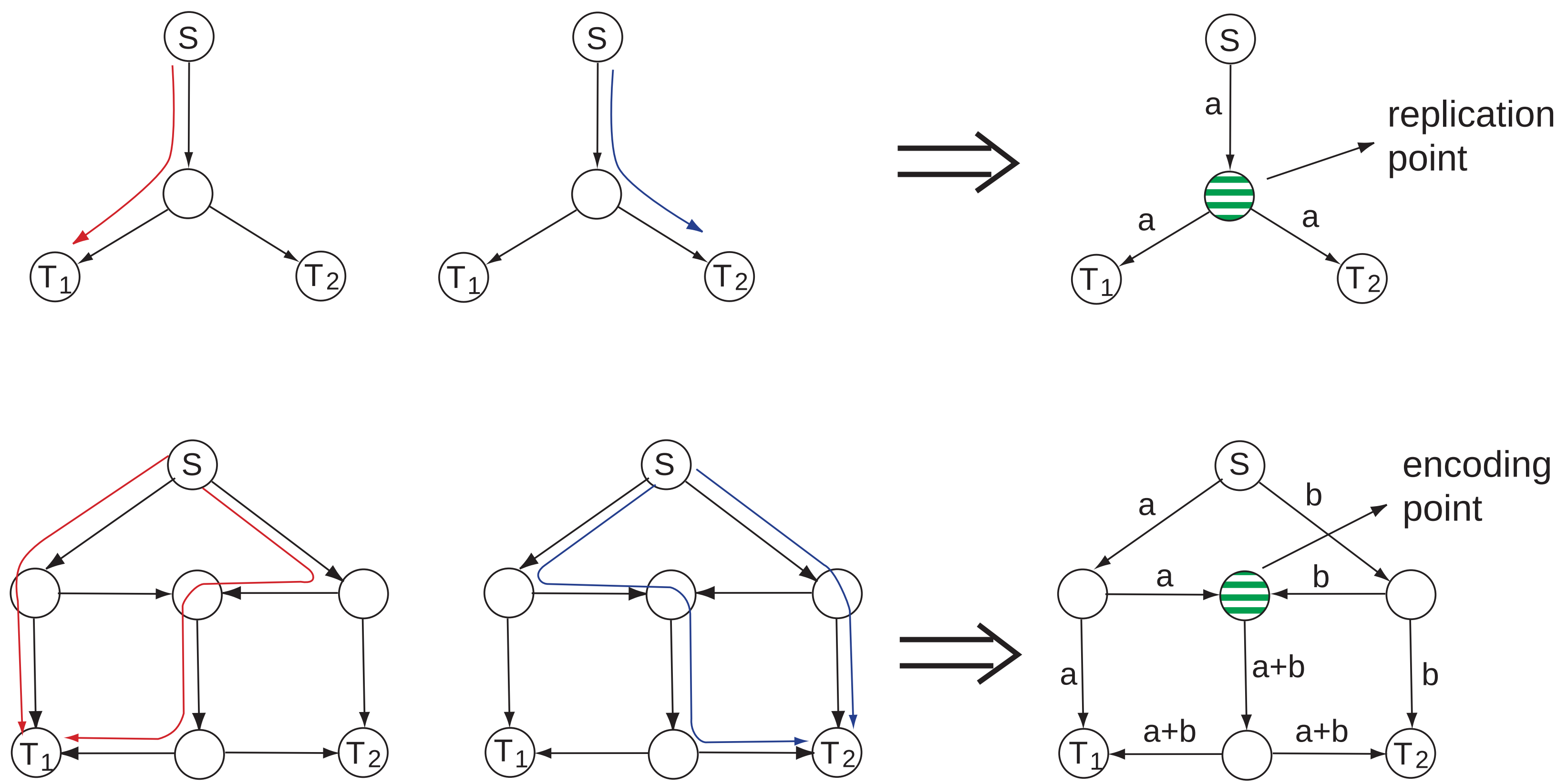
- Tang Xiahou, University of Calgary
- Chuan Wu, The University of Hong Kong

Talk Outline

1. Network Coding, Field Size
2. Examples in Planar Networks
3. The Sufficiency of $GF(3)$ in Planar Networks
4. The Sufficiency of Routing in Outerplanar Networks
5. The Case of Co-face Terminals
6. Conclusion and Open Problems

Multicast with Network Coding in Directed Networks

[ACLY'2000] A multicast rate h is feasible in a directed network if and only if it is feasible as a unicast rate to each receiver separately.



Some Basic Questions on Network Coding

- When is network coding necessary?
- How much benefit can network coding bring, over routing?
- How and where to encode, in general networks?
- The overhead of network coding?
- How large a field is required for coding?
- ......

Basic Questions on Network Coding

- When is network coding necessary?
- How much benefit can network coding bring, over routing?
- How and where to encode, in general networks?
- The overhead of network coding?
- How large a field is required for coding?

The answer often closely depend on the network configuration.

- network topology
- link capacity vector
- source/receiver location

Basic Questions on Network Coding

- When is network coding necessary?
- How much benefit can network coding bring, over routing?
- How and where to encode, in general networks?
- The overhead of network coding?
- **How large a field is required for coding?**

The answer often closely depend on the network configuration.

- network topology
- link capacity vector
- source/receiver location

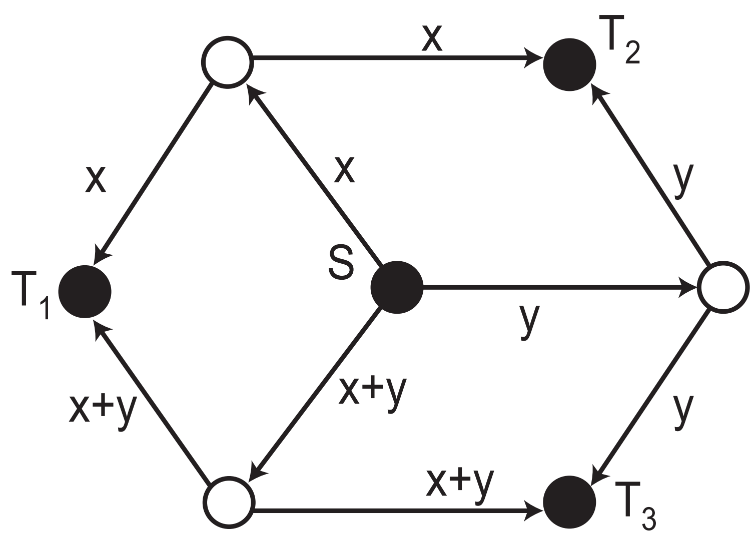
Small *vs.* Large Fields

- A large field makes code assignment easy: each receiver obtains linearly independent info flows.
- A small field leads to efficient encoding and decoding operations.
- A very small field may also lead to efficient code assignment algorithms.

Field Size Requirement

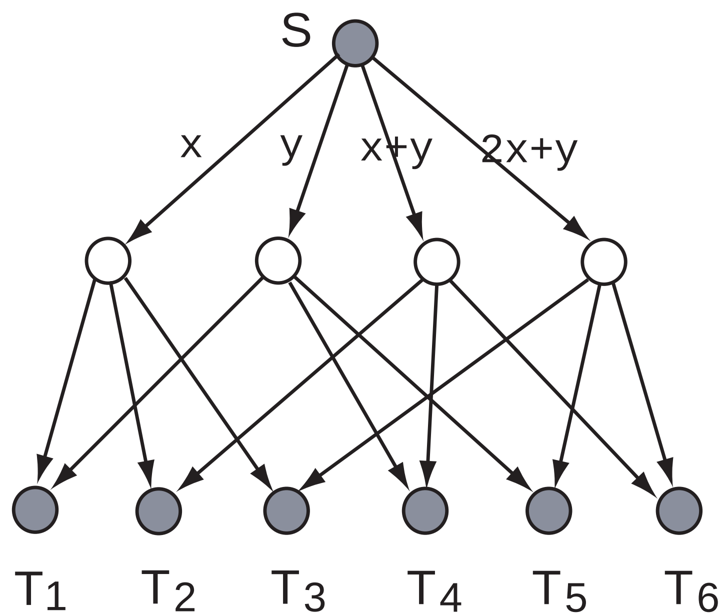
Let's focus on a single multicast session: one source, multiple receivers.

We need $GF(2)$ for $C_{3,2}$



Field Size Requirement

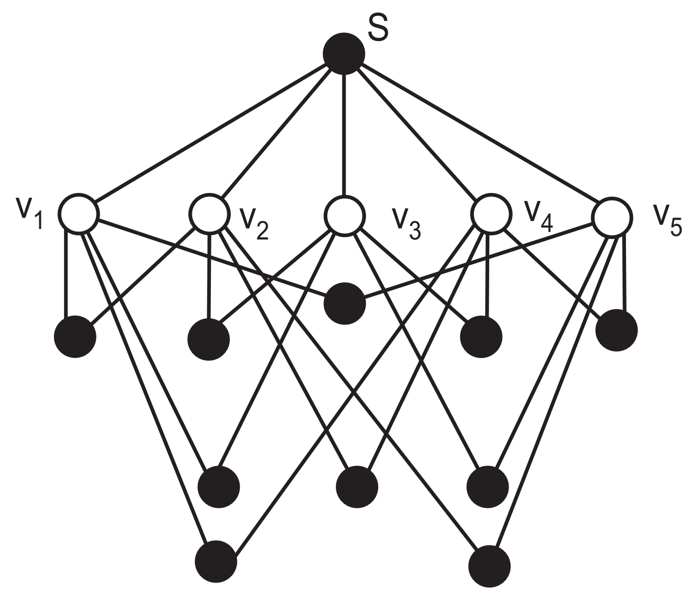
We need $GF(3)$ for $C_{4,2}$



Field Size Requirement

We need $GF(2^2)$ for $C_{5,2}$.

For $C_{n,2}$, we need $GF(q)$ where $q \geq n - 1$.



Current Picture

- Arbitrary networks: no field of constant size is always sufficient.
- Best known result: a field $GF(q)$ with $q \geq k$ is sufficient. (k: # of multicast receivers) (actually . . .)
- In practice: randomized network coding over $GF(2^8)$ or $GF(2^{16})$.

Main Message of This Talk

- Compared to an arbitrary network, a planar network is often a much better reflection of a network from practice.
  - Linear instead of quadratic number of links
  - Planar mesh topology instead of totally random connections.
- Small finite fields suffice for network coding in planar networks, and most practical networks.
- New deterministic code assignment algorithms
  - Requiring much smaller fields
  - Low (linear) or Moderate (quadratic) time complexity

Planar Graphs

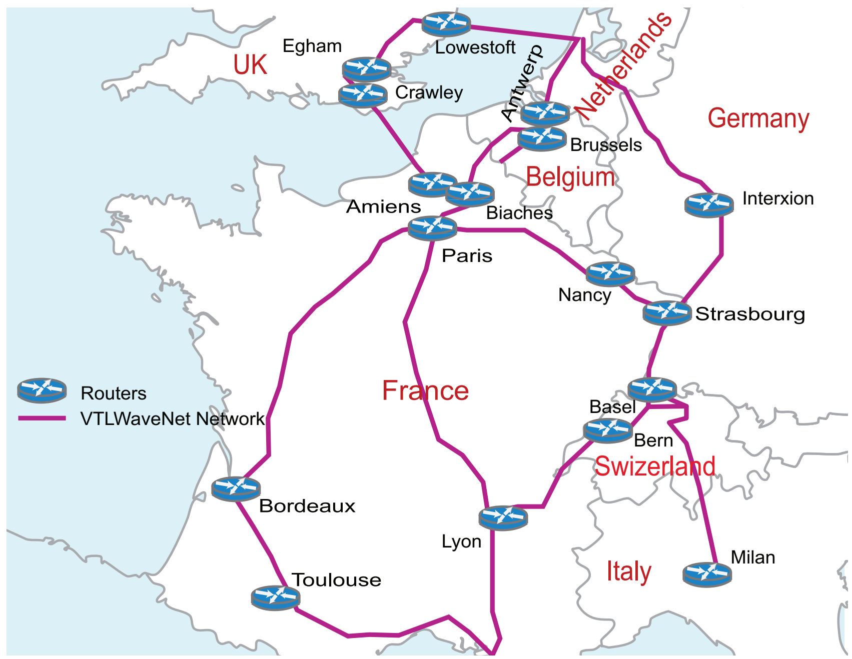
A planar graph is a graph that can be drawn in a 2-D plane without crossing edges.

Such a no-corssing drawing is called a planar embedding.

Planar graphs have many nice properties, and allow very efficient algorithms to be designed, for classic problems such as max flow, shortest path.

The VTLWaveNet in Europe

A real-world wide-area/backbone network, deployed along the surface of the globe, exhibits a natural planar embedding.

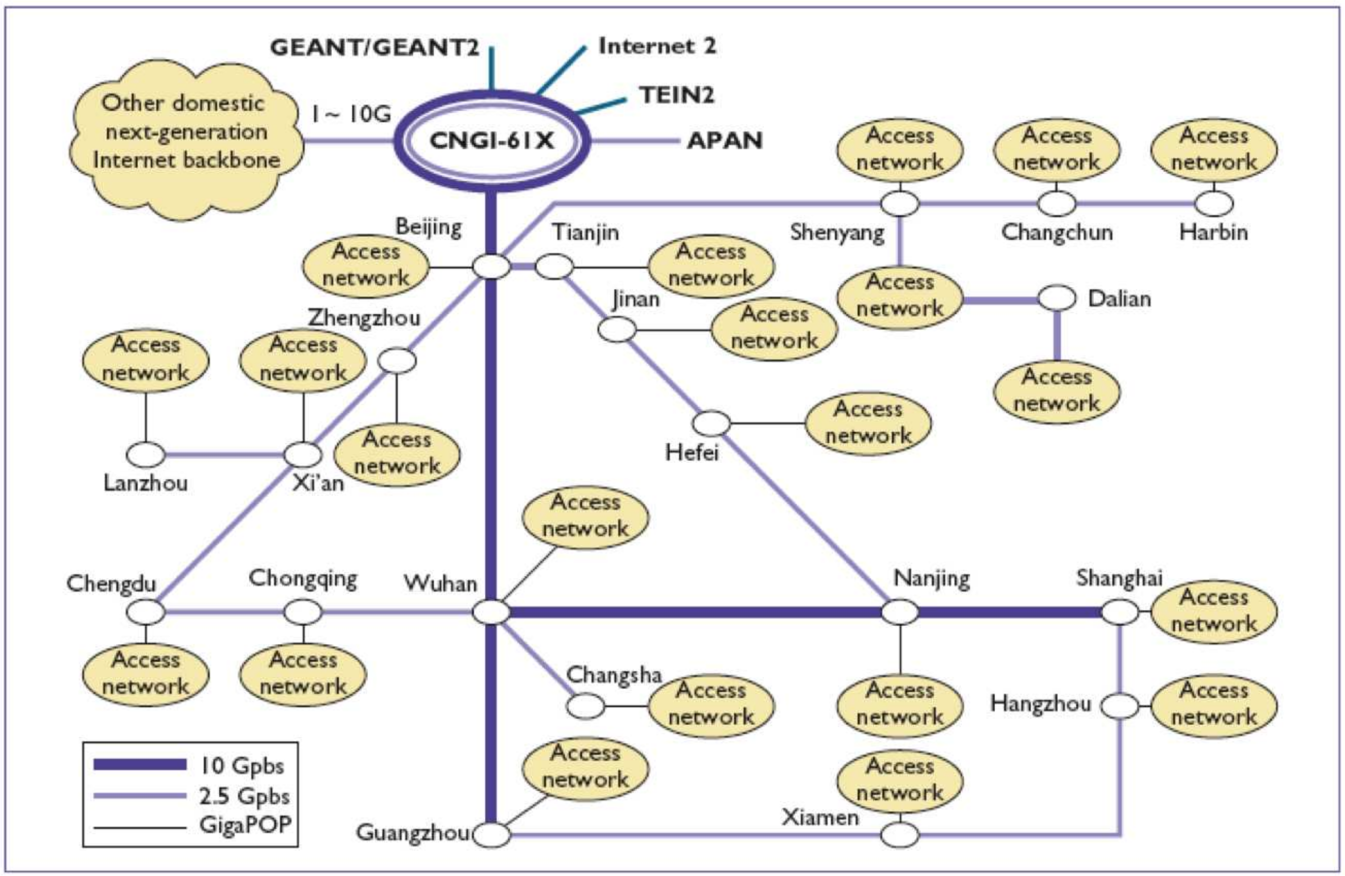


Tier-1 Optical Fiber Network in China

A canonical planar mesh network topology.

CERNET-2

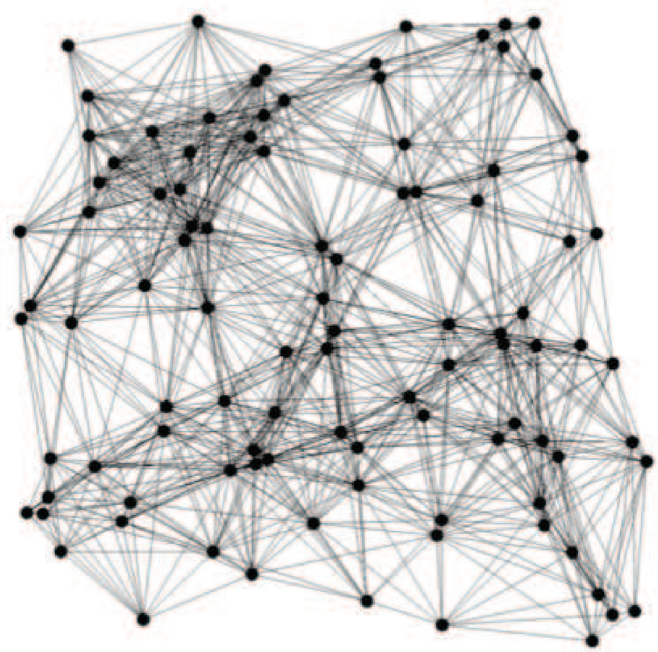
The IP-v6 network in China, courtesy of: Yong Cui @ Tsinghua



Realword Networks Far From Planar

A dense wireless sensor network.

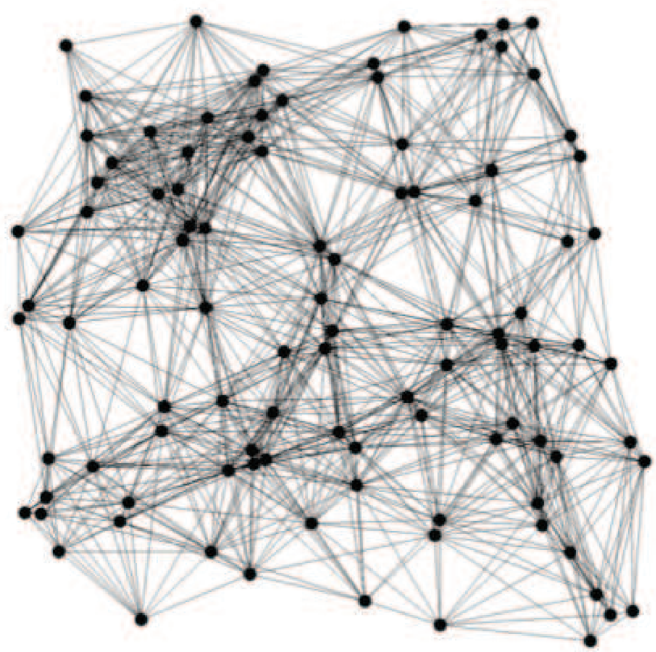
One of the most “non-planar” types of compute network.

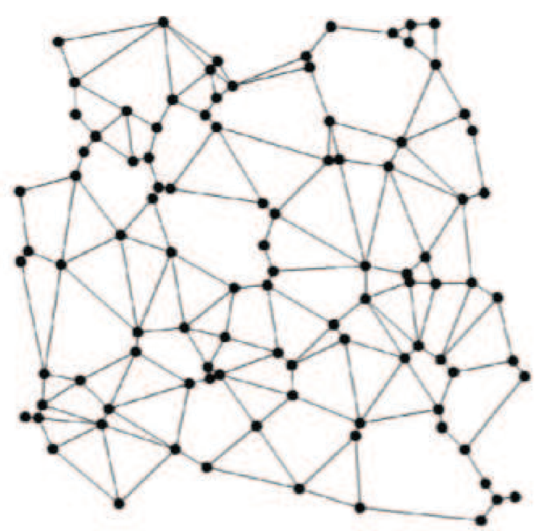


(example from [Alzoubi *et al.* 2003])

Planar Backbone of Dense Networks

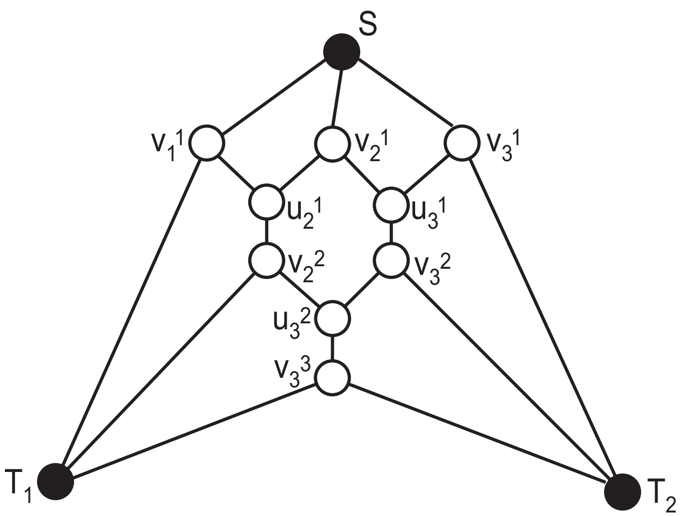
- Executing network protocols (broadcast, multicast *etc*) over a very dense network is extremely inefficient.
- A large series of work: extract a planar backbone, then run network protocols over the planar backbone.

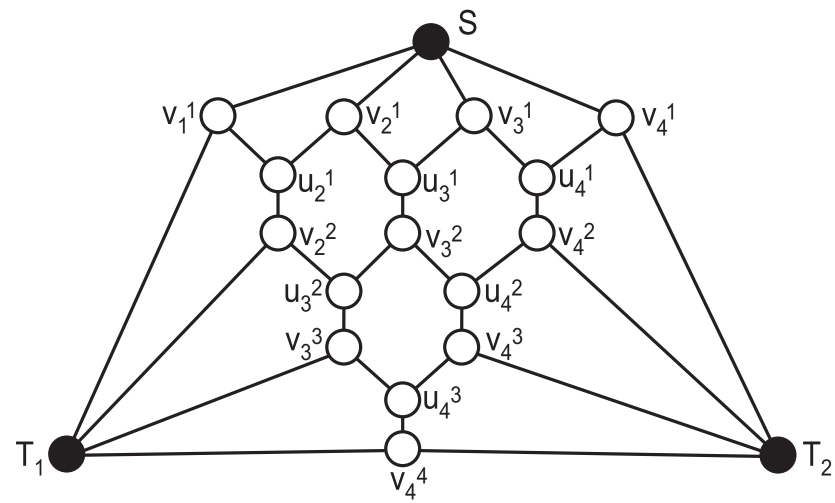




Example Planar Networks

- Requires coding at many nodes
- Yet coding over $GF(2)$ suffices.

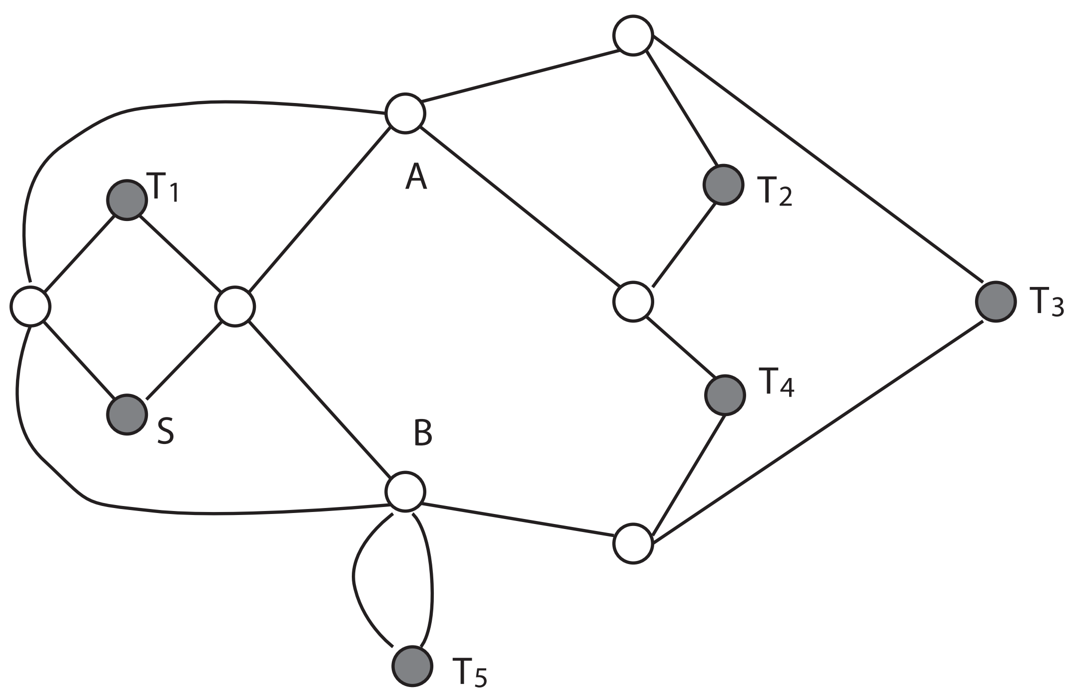




(example from [LSB 2006])

Another Example Planar Network

- A 'minimal' multicast network that requires network coding for multicasting two flows.
- Yet no particular node must perform encoding.
- Coding over $GF(2)$ suffices.



The Sufficiency of $GF(3)$ in Planar Networks

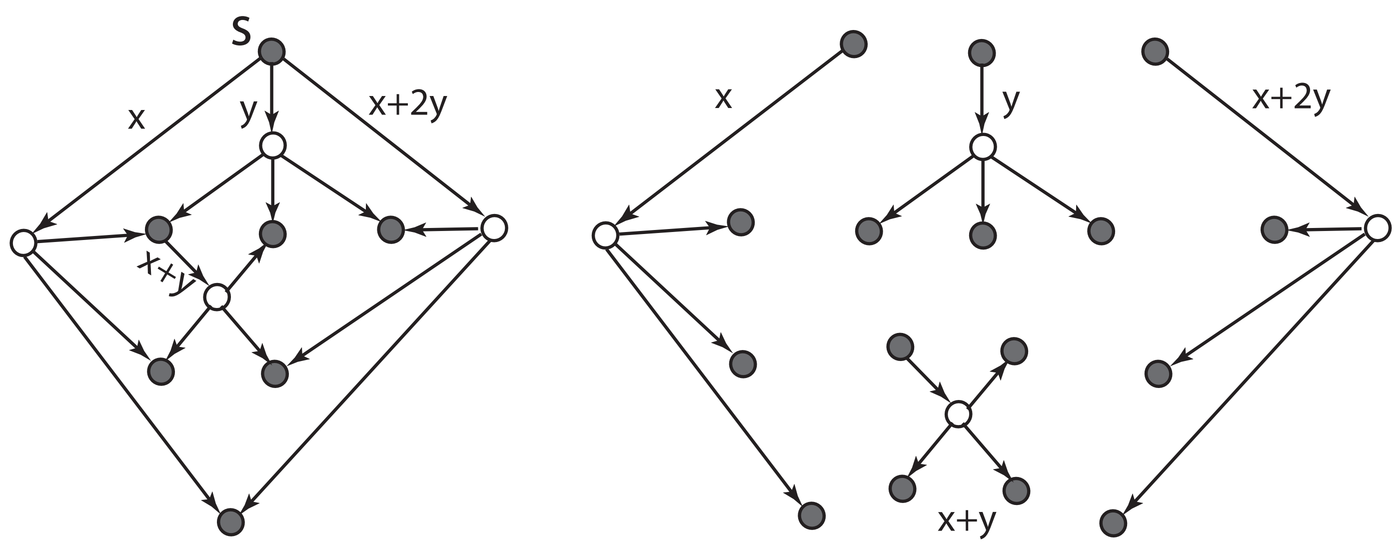
Theorem. For multicasting $h = 2$ flows in a planar network, coding over $GF(3)$ is sufficient.

Inspired by [FSS 2004] and [EGS 2006].

Conjecture: holds for any $h \geq 2$.

Sufficiency of $GF(3)$ — subtree decomposition

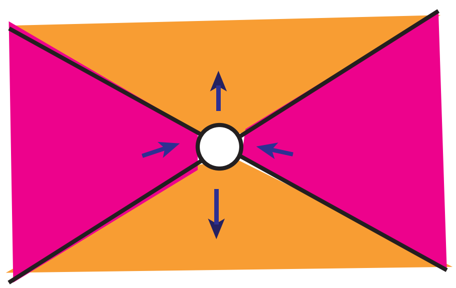
- Decompose a multicast flow into non-overlapping subtrees
- Each subtree has 1 root, ≥ 1 leaves
- Each root has in-degree 2

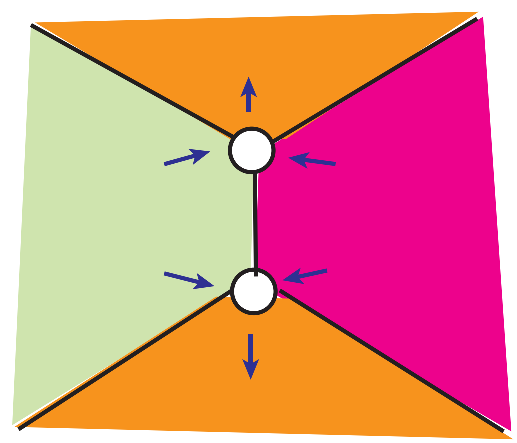


(A planar bipartite network that 'mimics' $C_{4,2}$, and hence requires $GF(3)$.)

Sufficiency of $GF(3)$ — node expansion

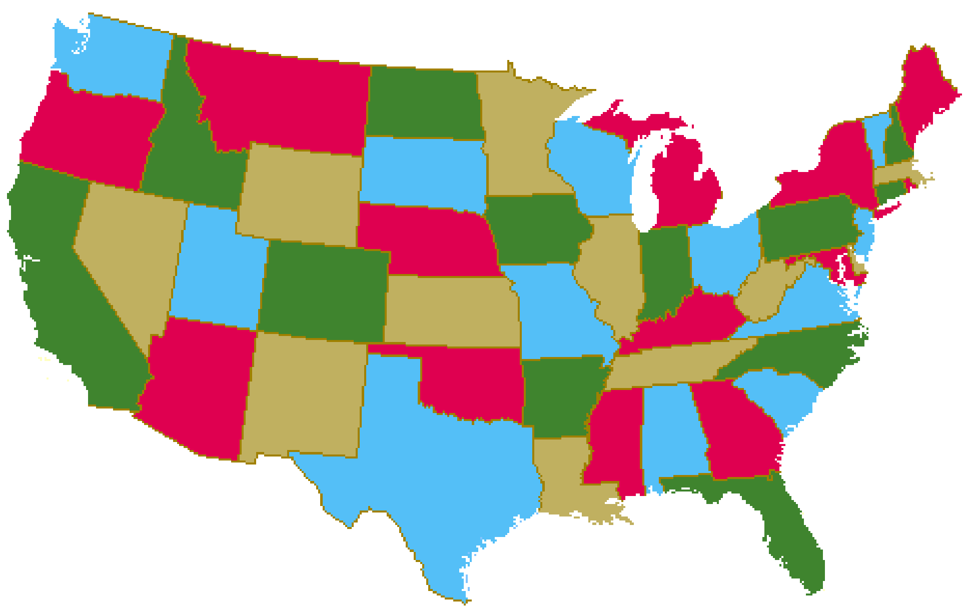
- Decompose the plane into faces, each containing one subtree
- If a node has two opposite faces ‘feeding into’ it, perform expansion
- Prepare for four-coloring a planar network





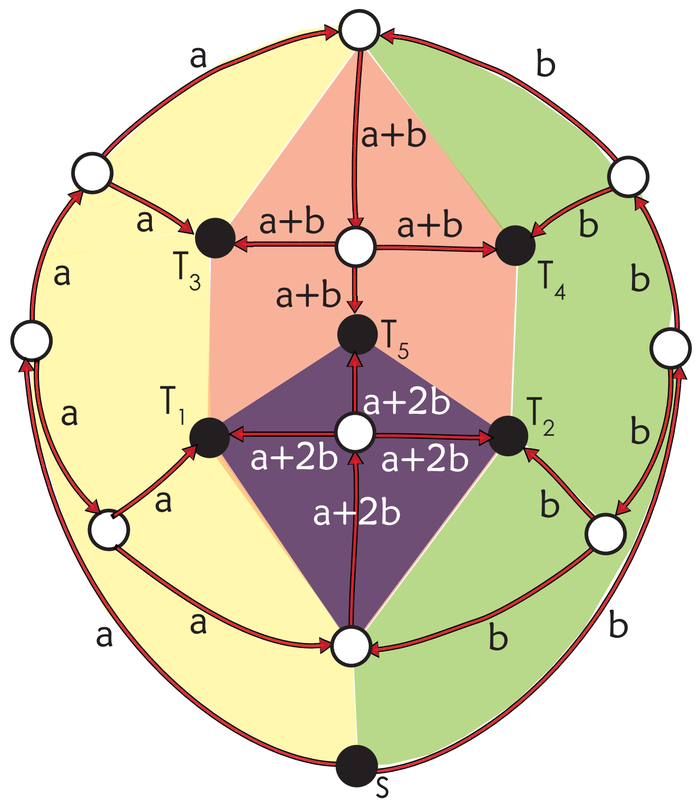
Sufficiency of $GF(3)$ — 4-coloring a planar graph

- Every planar graph is 5-colorable ([Kempe 1879]), and such coloring can be done in $O(n)$ time ([CNS 1981]).
- Every planar graph is 4-colorable ([Appel & Haken 1976]), and such coloring can be done in $O(n^2)$ time ([RSST 1996]).



Sufficiency of $GF(3)$ — four-coloring a planar graph

- Code assignment over $GF(3)$.
- The four colors: x, y, $x+y$, $x+2y$.



(Another planar multicast network requiring $GF(3)$.)

$GF(3)$ vs. $GF(2^2)$

$GF(2^2)$ may be preferred over $GF(3)$ in practice, for:

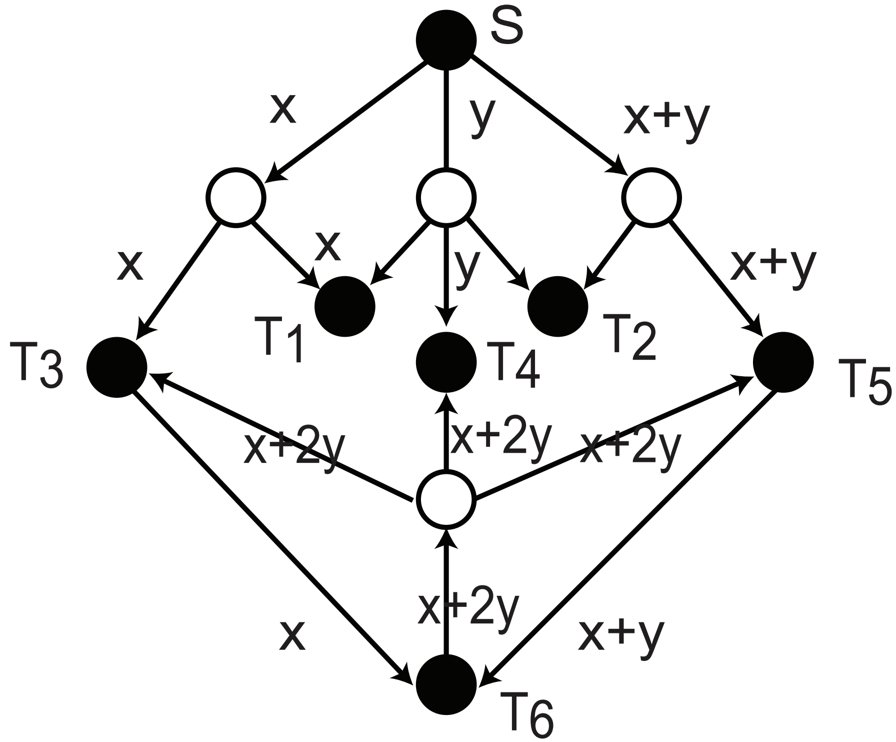
- '$+$' between two packets/flows is simply bit-wise xor
- Code assignment complexity is $O(n)$ instead of $O(n^2)$
- 2-bit symbol representation without wasted symbols

Randomized Network Coding in Planar Networks

- Randomized network coding has the same $O(n)$ complexity as 5-coloring.
- Appears incapable of exploiting planarity.
- Success probability of randomized code assignment for multicast in random planar networks:

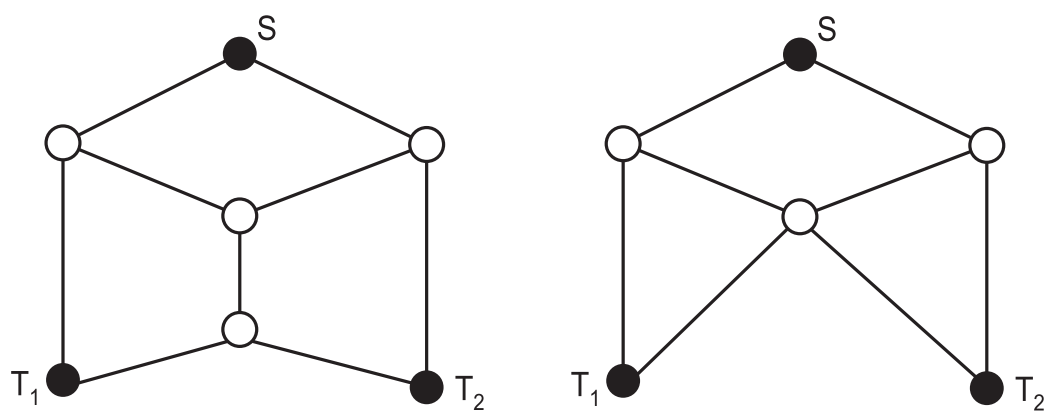
field size	2	3	5	7	11	23	131	311
success rate	0.296	0.423	0.582	0.670	0.770	0.881	0.979	0.991

Necessity of $GF(3)$ - Example # 3



Outerplanar Multicast Networks

- Outerplanar: all nodes adjacent to a common face
- Contracting the bottleneck link in the butterfly network leads to an outerplanar network
- Network coding not necessary anymore



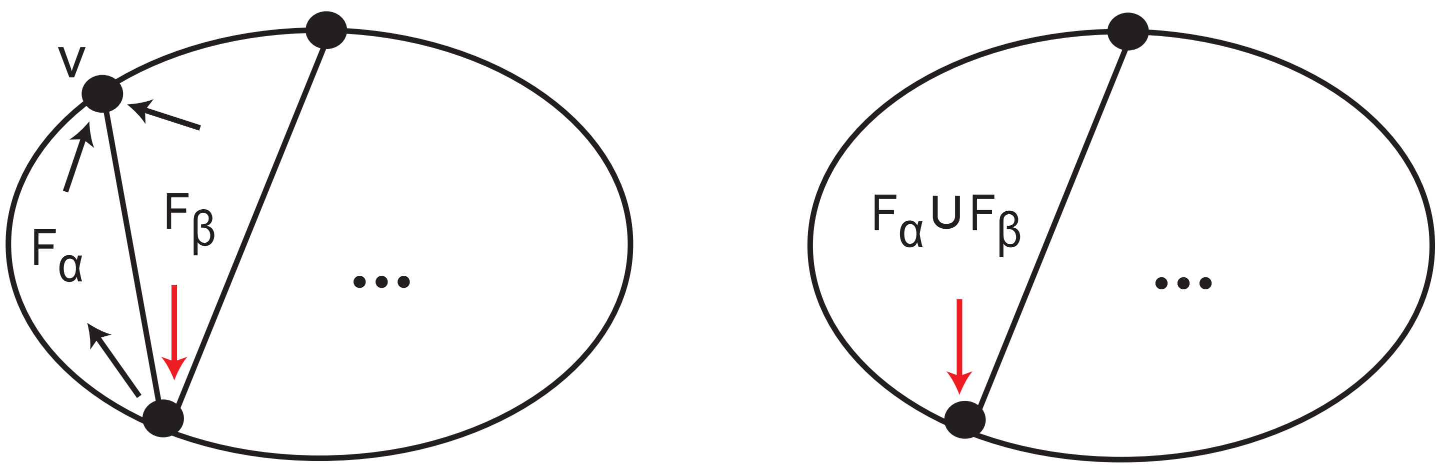
Outerplanar Multicast Networks

Theorem. Network coding is equivalent to routing in an outerplanar network, for $h = 2$.

Conjecture: holds for any $h \geq 2$.

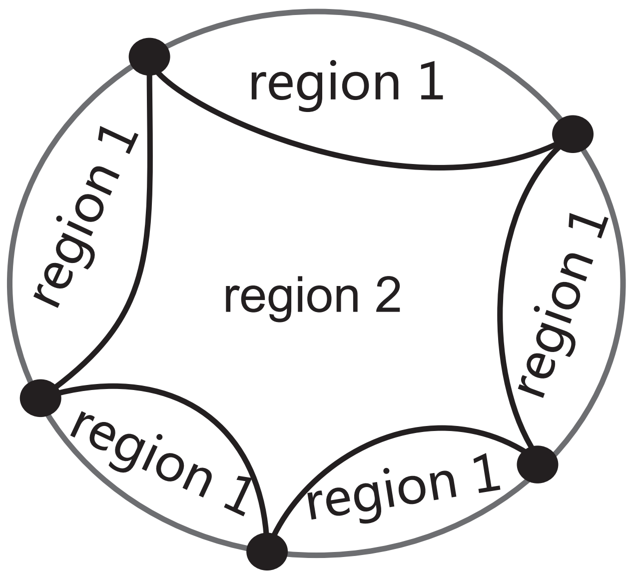
Outerplanar networks — face merging

- Subtree decomposition, as usual.
- Face merging.



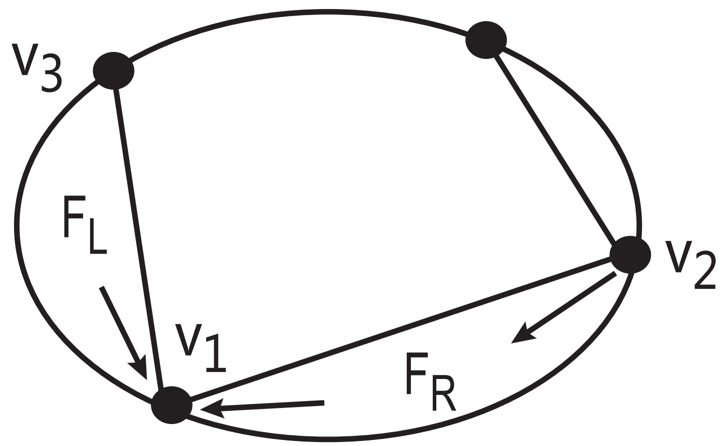
Outerplanar networks — two types of regions

- Region 1: boundary faces.
- Region 2: the inner region.



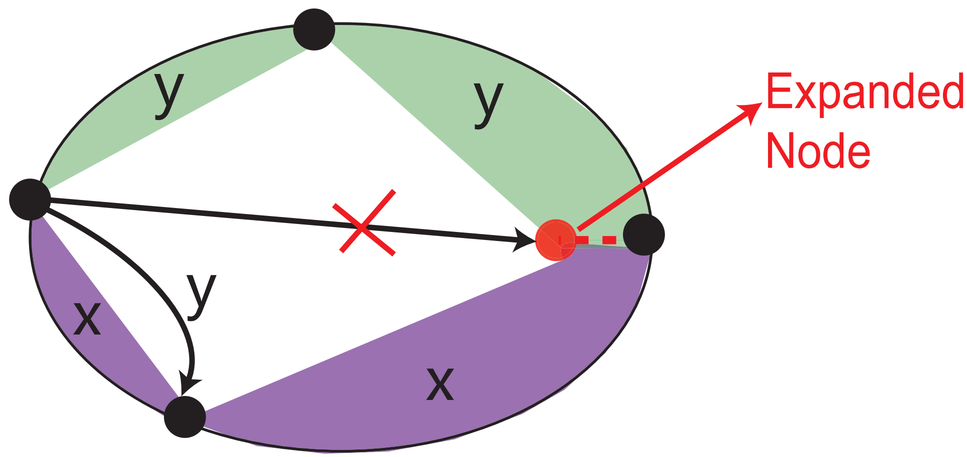
Outerplanar networks — coloring region 1

- Coloring faces in region 1, using two colors only.



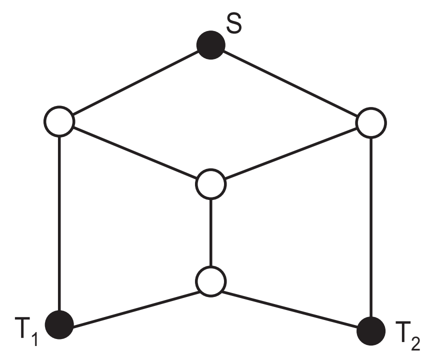
Outerplanar networks — coloring region 2

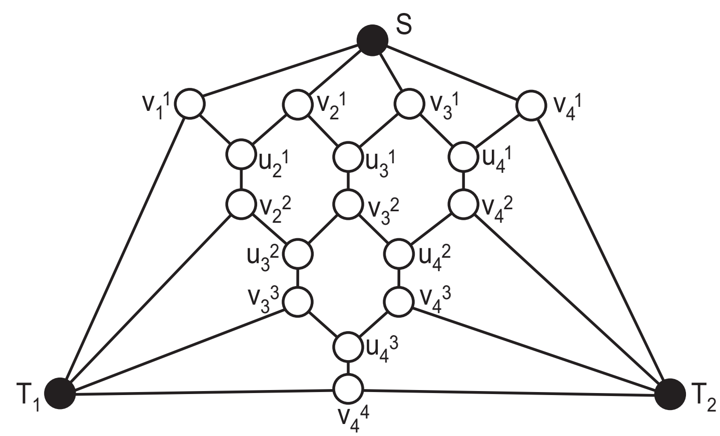
- Coloring chords in region 2, one at a time, without using a third color.



The Case of Co-face Terminals

The case between planar and outerplanar: all multicast terminals lie on a common face.





Conjecture: Coding over $GF(2)$ is sufficient in this case.

Conclusion

We proved that, for multicasting $h = 2$ flows:

- $GF(3)$ is sufficient for general planar networks.
- Routing is sufficient for outerplanar networks.

Future Work

We conjecture that, for multicasting any $h \geq 2$ flows:

- $GF(3)$ is sufficient for general planar networks.
- $GF(2)$ is sufficient for terminal co-face networks.
- Routing is sufficient for outerplanar networks.

A Graph Minor Perspective to Network Coding

We conjecture that:

- If a directed multicast network G requires network coding for achieving maximum throughput, then G contains a K_4 minor.
- If an udirected multicast network G requires network coding for achieving maximum throughput, then G contains a $C_{3,2}$ minor.
- If a multicast network G requires $GF(2^2)$ for achieving maximum throughput, then G contains a K_5 minor.
- There exists a function $f(q)$, such that if a multicast network G requires $GF(q)$, then G contains a $K_{f(q)}$ minor, and $f(2) = f(3) = 4$, $f(4) = 5$.