The Encoding Complexity for Network Coding with 2 Simple Multicast Sessions

Kai Cai

Institute of Computing Technology,
Chinese Academy of Sciences

Joint work with Wentu Song, Ronguan Feng,
Peking University

Outline

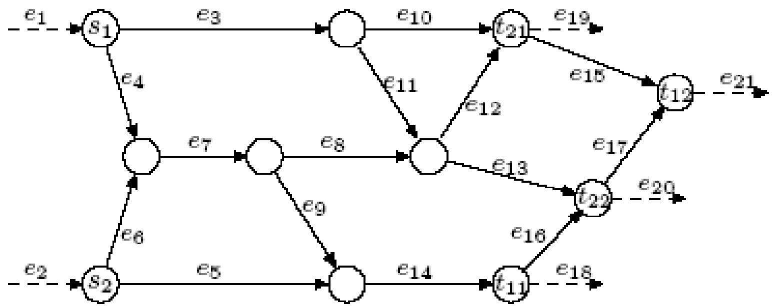
- Introduction
- The Method --- Region Decomposition
- Time Complexity
- Encoding Links
- Encoding Field

Introduction

2 Simple Multicast Sessions

- A finite, directed, acyclic graph (V, E).
- Links: unit capacity, delay free, error free.
- Two source nodes each generate a unit rate message.
- Each message is demanded by a set of sinks, source≠sink.
- The message is regarded as random variable taken values from some finite field F, i.e., the encoding field.

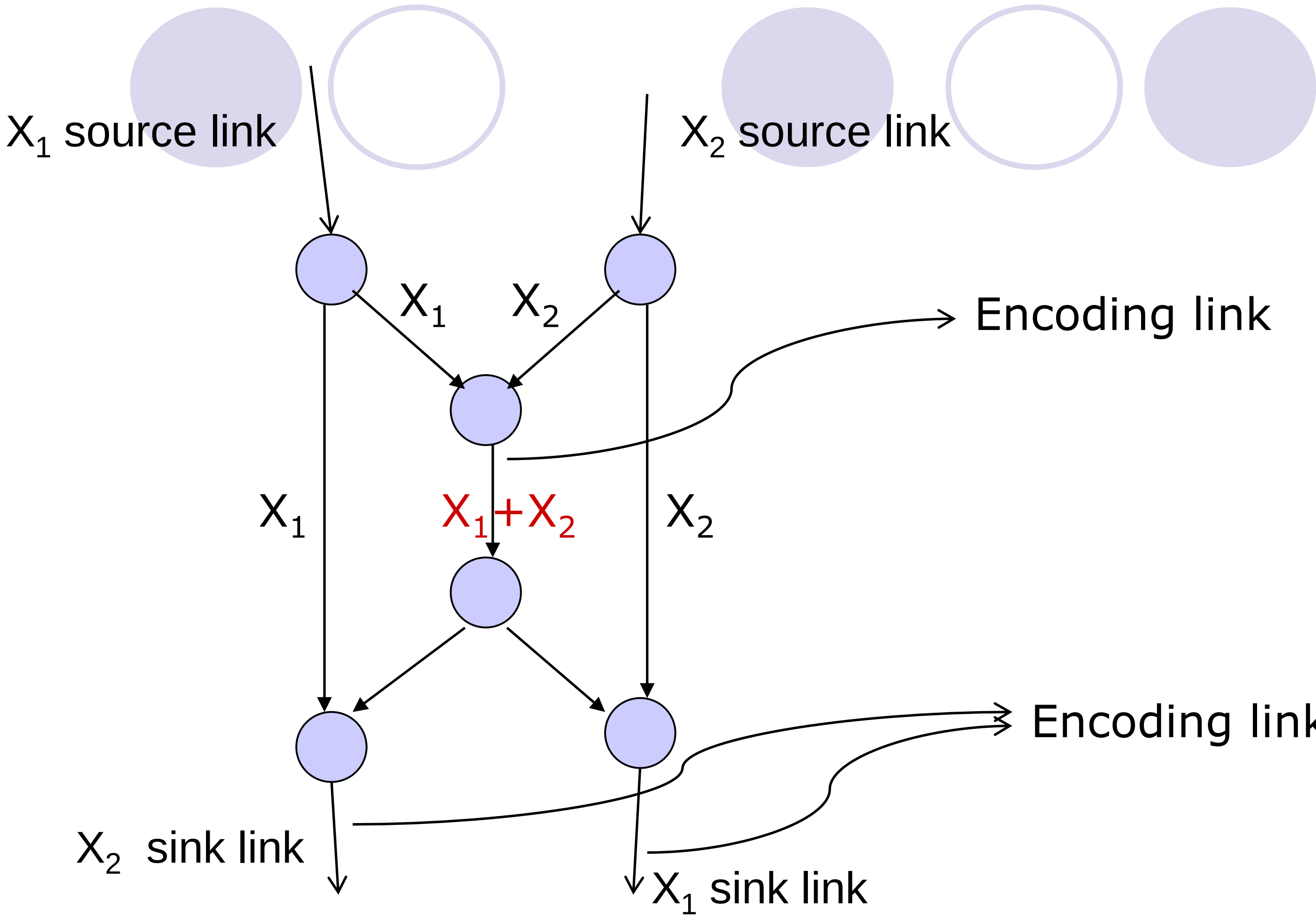
- A 2 simple multicast network with 4 sinks.



- Source nodes s_1 s_2 generate x_1 x_2, respectively.
- x_1 and x_2 are demand by $t_{1,1}t_{1,2}$ and $t_{2,1}t_{2,2}$ respectively.
- We add an imaginary link to each source node (say x_i source
- link) and an imaginary link to each sink node (say x_i sink link), i=1,2.

Our Problems

- Does a linear solution exist?
- What is the time complexity to obtain a solution?
- How many encoding links is sufficient to obtain a solution?
- What is the required size of the finite field for a solution?


X1 source link
X2 source link
X1
X2
Encoding link
X1
X1+X2
X2
Encoding link
X2 sink link
X1 sink link

Known Results

- ✓ The solvability can be determined in polynomial time.
- ✓ A solution can be obtained in polynomial time.

- ✓ The order of the time complexity; the number of encoding links; the required field size (Not yet)

C.-C. Wang and N. B. Shroff, ``Pairwise Intersession Network Coding on Directed Networks,'' IEEE Trans. Inf. Theory, vol. 56, no. 8, pp. 3879-3900, Aug. 2010.

S. Fortune, J. Hopcroft, and J. Willie, ``The directed subgraph homeomorphism problem,'' Theoretical Computer Science, vol. 10. pp. 111-121, 1980.

Our Results

- ✓ The solvability can be determined with time O(|E|).
- ✓ A solution can be obtained with time O(|E|).
- ✓ max{3, 2N-2} encoding links is sufficient to achieve a solution.
- ✓ A finite field with size $\max\{2, \lfloor\sqrt{2N-7/4}+1/2\rfloor\}$ is sufficient to achieve a solution.

Here, |E| is the number of links and N is the number of sinks of the underlying network.

The method

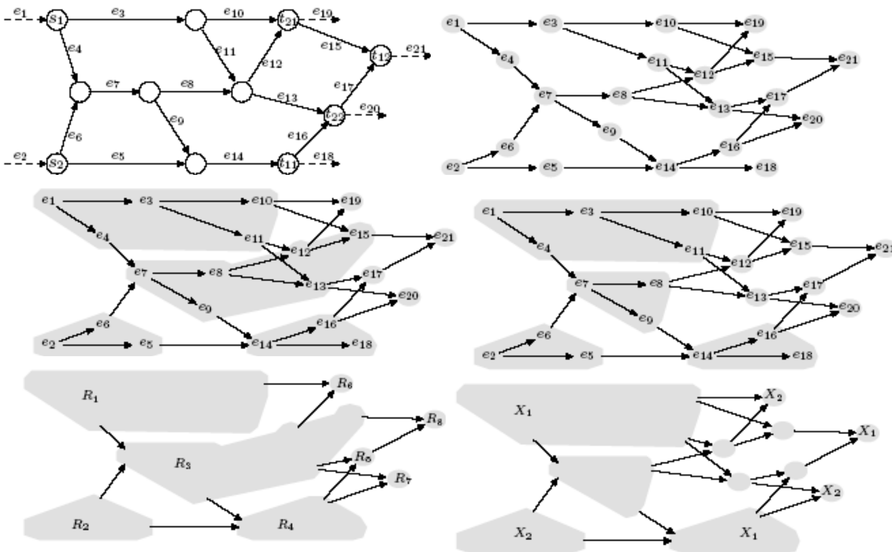
Region Decomposition

It is a promotion of the subtree decomposition method for multicast networks: C. Fragouli and E. Soljanin, "Information flow decomposition for network coding," IEEE Trans. Inf. Theory, vol. 52, no. 3, pp. 829-848, Mar. 2006

Region Decomposition and Region Graph

- A region is a collection of links, namely R such that except one of them (called the head of R), each has an incoming link in R.
- A region decomposition is a partition of the link set of mutually disjoint regions $D=\{R_{1,} R_{2,} \ldots, R_k\}$.
- A region graph RG(D) with respect to D is a graph with node set $D=\{R_{1,} R_{2,} \ldots R_k\}$ and two regions is adjacent if a link in one region is adjacent to the head of the other region.

Examples



Remarks

- ✓ The line graph L(G) is a (trivial) region graph.
- ✓ All the region graphs can be constructed from L(G) by combining adjacent regions again and again.

Codes on the Region Graph

- A code on a region graph RG(D) is a collection of 2-dimensional vectors assigned to D={R_1, R_2, …R_k} such that:

(1) If R contains an X_1 (source or sink) link, then assign (1,0).
If R contains an X_2 (source or sink) link, then assign (0,1).

(2) for each non-source region R, the vector assigned to R is a linear combination of the vectors of its parents.

If a code exists, we call the region graph feasible.

Basic idea: assign a same global encoding kernel to the links in the same region.

Remarks

- G is solvable if and only if L(G) is feasible.
- G is solvable if and only if it has a feasible region graph.
- Suppose RG(D) is feasible, the following operations do not change the feasibility of RG(D):

(1) If R has a single parent P, then combine R with P.

(2) If two adjacent regions P and R are assigned with same vectors, then combine P and R.

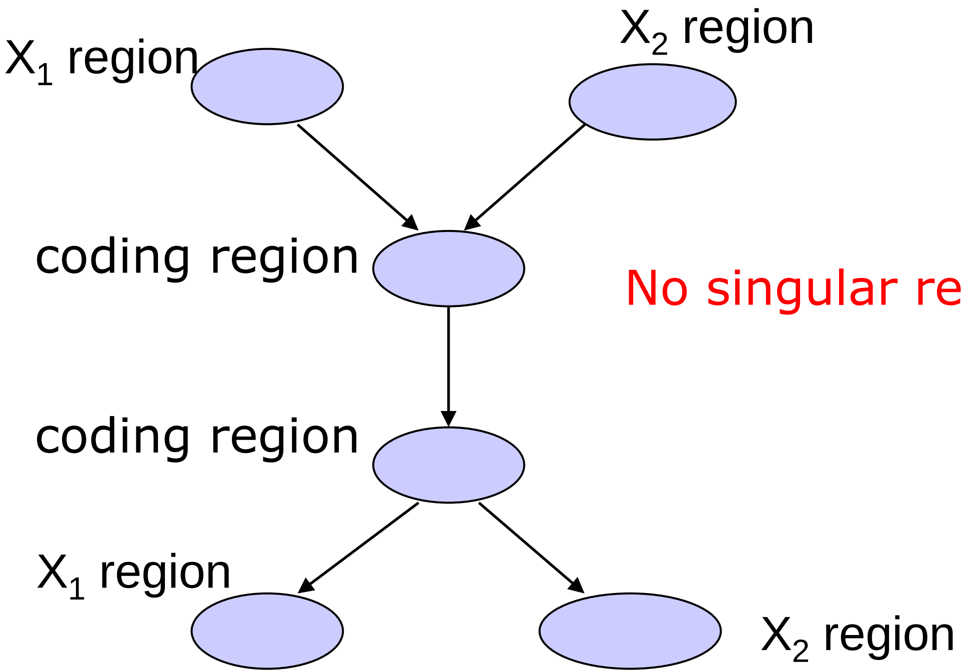
Region labeling

- If R contains an X_1 (source or sink) link, then label $x_{1.}$
- If R contains an X_2 (source or sink) link, then label x_2
- If all the parents of R are all labeled with x_i are labeled R with x_i , for i=1,2.

Notations

- *x_i region*: R is labeled with x_i , for i=1,2.
- *coding region*: R is labeled neither x_1 nor x_2.
- *singular region*: R is labeled both x_1 and x_2.

Obviously, If D is feasible then D has no singular region, but the other direction is not true in general.



No singular region but infeasible!

- Result:

✓ Suppose D has no singular region. If each non-source region of D has at least two parents, then D is feasible.

✓ Proof.

✓If D satisfied the condition, then we can decentralized assign the global encoding kernels, i.e., assign mutually linear independent vectors {(1,0), (0,1), (1, a_1), (1, a_2), …(1, $a_{|F|-1}$)} respectively to the X_1 regions, X_2 regions and all the coding regions of D. Here, $F=\{0,1=a_1, a_2, \ldots, a_{|F|-1}\}$ is the encoding field.

The Time Complexity

- Basic Region Decomposition
  - A region decomposition D** satisfies:

    (1) Each non-source region has at least two parents;

    (2) Except the head, all the incoming links of a link are within the same region.

- Remarks:
  - (a) G has a unique basic region decomposition.
  - (b) D** can be obtained with time O(|E|).

Algorithm **1**: *Region Decomposing* $(G = (V, E))$

$R_1 = \{e_1\};$

$R_2 = \{e_2\};$

$K = 2;$

$j = 3;$

While $j \leq |E|$ **do**

  if there is a $k \in \{1, \cdots, K\}$ such that $In(e_j) \subseteq R_k$ **then**

    $R_k = R_k \cup \{e_j\};$

  else

    $K = K + 1;$

    $R_K = \{e_j\};$

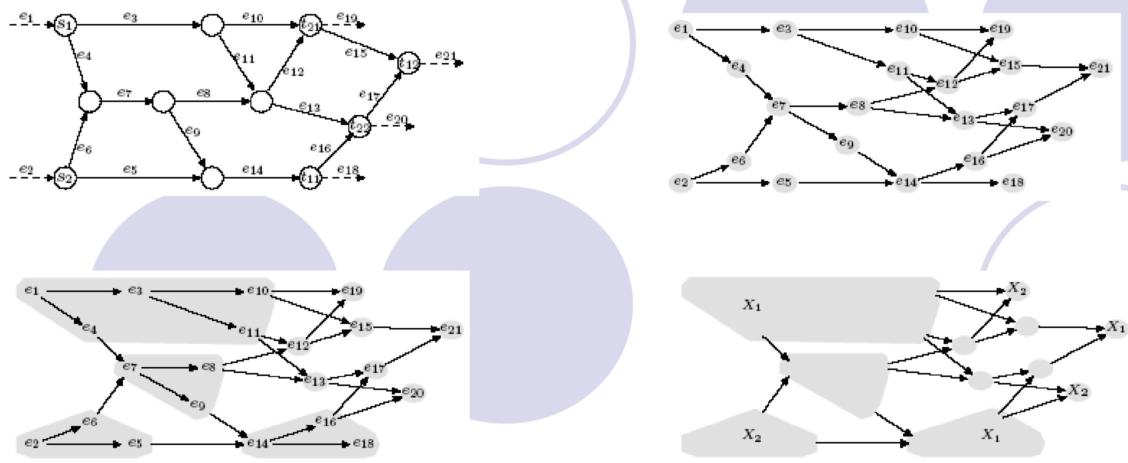
  end if

  $j = j + 1;$

end while

return $D^{**} = \{R_1, \cdots, R_K\};$

An Example

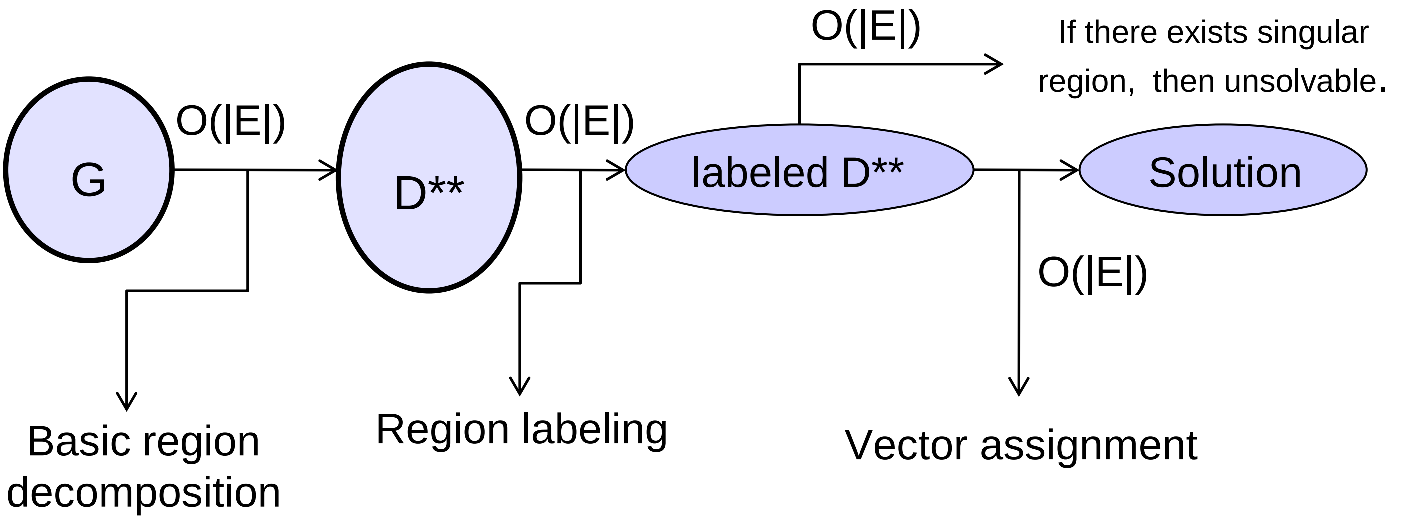


e1
e2
e3
e4
e5
e6
e7
e8
e9
e10
e11
e12
e13
e14
e15
e16
e17
e18
e19
e20
e21
s1
s2
t21
t12
t22
t11
X1
X2

Main Results:

- G is solvable if and only if D** has no singular region.
- The solvability of G can be determined with time O(|E|);
- A linear solution of G can be obtained with time O(|E|).

How to determine the solvability and/or achieve a (linear) solution?



The Encoding Links

- Minimal Feasible Region Graph

✓ A minimal feasible region graph RG(D) satisfies:

(1) Contraction of any adjacent regions results in infeasible;

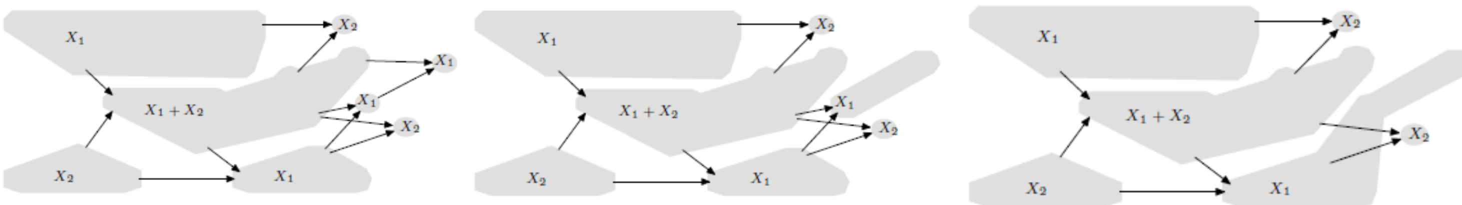
(2) Deleting any link of RG(D) results in infeasible.

- Remarks:

✓ (a) A minimal feasible region graph can be obtained from any feasible region graph by region contraction and edge deletion again and again till (1) (2).

✓ (b) A minimal feasible region graph has the smallest number of encoding links and also needs the smallest encoding fields.

An Example



(a) (b) (c)

Results on MFRG

- 1.1 Each non-source region has exactly 2 parents. (If one parent, then it can be combined with his parent ; If more than two parents , then we can delete links, noticing that dim=2.)

- 1.2 Two regions which are adjacent or having a common child can not be both x_1 regions or x_2 regions. (If two x_1 region are adjacent, then we can combine them; If they have a common child, then we can delete 1 link.)

- 1.3 Two adjacent coding regions has a common child. (If no common child, then we can combine the two coding regions, noticing that 1.1 and decentralized assignment of vectors.)

- 1.4 If a coding region R is adjacent to an x_1(or x_2) region, then R has a common child with some other x_1(or x_2) region P. (otherwise, R have no common child with any x_1(or x_2) region, then we can combine R with the x_1(or x_2) region.)

Results on MFRG

- 2.1 An x_i region is either an x_i source or x_i sink region, i=1,2. (by 1.2)
- 2.2 A coding region has at least two children of sink regions. (by 1.1. For an x_i parent, by 1.4, we can finally construct one; For a coding parent, by 1.3, we can finally construct one.)
- 2.3 If R is a coding region having no child of coding region, then R has two children of x_1 and x_2 region such that the x_i region has an x_j parent, $i \neq j$. (by 2.2, we find an x_i child first, then by 1.4, R has a common child Q with some x_j region, by the assumption, Q is an x_j child ($i \neq j$). Consider x_j, again by 1.4, we have an x_i child which has an x_j parent.)

Main Result and the Idea

- If the number of sinks $N \geq 3$, then $2N-2$ encoding links is sufficient to achieve a network coding solution. $\Leftrightarrow$ For a MFRG with $N \geq 3$, the number of the coding regions $n \leq N-2$.

- Note that In a MFRG, a link is an encoding link if and only if it is the head of a coding region or a sink region.

- Estimation of the number of the coding regions.

Main Idea of the Proof

- Estimate J : the number of edges between a coding region to a sink region.
- Suppose P and Q are coding regions with *biggest* indexes. Two cases:

  (1) Q is a child of P;

  (2) P and Q are not adjacent.

Case (1): $J \leq 2N-2$ (by 1.1, 2.3) ; $J \geq 2n+1$ (by 2.2,1.3).

Case (2): $J \leq 2N-4$ (by 2.3) ; $J \geq 2n$ (by 2.2).

Combine the two inequalities, respectively, we have $n \leq N-2$.

The Field Size

- Basic idea

✓ For a finite field F with size q, there exist q+1 mutually linearly independent vectors {(1,0), (0,1), (1, a_1), (1, a_2) ,…, (1, $a_{|F|-1}$)} in F^2.

✓ Assign (0,1), (1,0) to X_1 regions and X_2 regions respectively and two linear independent vectors to two coding regions which have a common child.

Associate graph

- Suppose the MFRG has n coding regions $Q_1, Q_2, \ldots Q_n$. The associate graph has n+2 vertexs $X_1, X_2, Q_1, Q_2, \ldots, Q_n$ and the following three kind of edges:
- (X_1, X_2);
- (Q_i, Q_j) if Q_i, Q_j has a common child;
- (X_i, Q_j) if X_i, Q_j has a common child.

- Estimate the chromatic number k of the associate graph (a field with size k-1 is sufficient to achieve a solution).

Lemmas

- Lemma 1: The X_1 source region the X_2 source region has a common child.
  (consider the first coding region, by 1.1)
- Lemma 2: Every vertex of the Associate Graph has degree at least 2.
- Lemma 3: Every k-chromatic graph has at least k vertices of degree at least k-1.

Proof of Lemma 2

- X_1 region have a common child with X_2 region and also with the maximal coding region (by 2.3).
- X_2 region have a common child with X_1 region and also with the maximal coding region.
- Coding region R have no coding child. Then (R, X_1) and (R, X_2) (by 2.3).
- Coding region R have a coding child Q. Then (P, Q) (by 1.3) and some (P, X_i) (by 2.2, 1.4).

- Main Idea of the Proof

✓ Estimate J : the number of edges of the associate graph.

$$J \geq \left[k(k-1)+2(n+2-k)\right] / 2$$ (By Lemmas 2, 3)

$$J \leq N+n$$ (By 1.1)

Combine the two inequalities, we obtain $k \leq \sqrt{2N-7/4}+3/2$.

Thanks ! and Questions?