Compute-and-Forward Network Coding Design over Multi-Source Multi-Relay Channels

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Multi-Source Multi-Relay System Channel

1. Lattice Codes

Definition 1 (Lattice): An n-dimensional lattice $\Lambda \subset \mathbb{R}^n$, is a set of infinite points in $\mathbb{R}^n$ such that if any two points λ_1, $\lambda_2 \in \Lambda$, then $\lambda_1 + \lambda_2 \in \Lambda$; if $\lambda_1 \in \Lambda$, then $-\lambda_1 \in \Lambda$. Every lattice can be written in terms of a lattice generator matrix $\mathbf{G} \in \mathbb{R}^{n \times n}$,

$$\Lambda = \{\lambda : \lambda = \mathbf{G}\mathbf{c}, \mathbf{c} \in \mathbb{Z}^n\}. \tag{1}$$

Definition 2 (Quantizer): A lattice quantizer is a mapping $\mathbf{Q}_\Lambda : \mathbb{R}^n \to \Lambda$, that sends a point $\mathbf{x} \in \mathbb{R}^n$, to the nearest lattice point $\lambda \in \Lambda$ in Euclidean distance,

$$\mathbf{Q}_\Lambda(\mathbf{x}) = \mathit{arg} \min_{\lambda \in \Lambda} ||\mathbf{x} - \lambda|| \tag{2}$$

Definition 3 (Voronoi Region): The Voronoi region $\mathbb{V}_\Lambda(\lambda)$ of a lattice point $\lambda \in \Lambda$ contains all the points $\mathbf{x} \in \mathbb{R}^n$ closest to λ in Euclidean distance,

$$\mathbb{V}_\Lambda(\lambda) = \{\mathbf{x} : \mathbf{Q}_\Lambda(\mathbf{x}) = \lambda\}. \tag{3}$$

The fundamental Voronoi region of a lattice Λ is the set of all points $\mathbf{x} \in \mathbb{R}^n$ that are closest to the origin,

$$\mathbb{V}_\Lambda = \mathbb{V}_\Lambda(\mathbf{0}) = \{\mathbf{x} : \mathbf{Q}_\Lambda(\mathbf{x}) = \mathbf{0}\}. \tag{4}$$

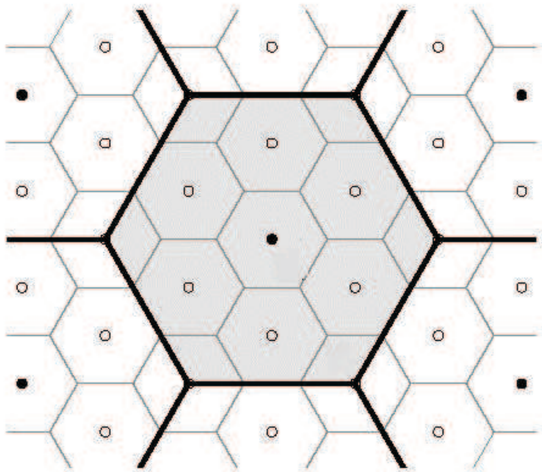
Definition 4 (Modulo): The modulo operation of $\mathbf{x} \in \mathbb{R}^n$ regarding a lattice Λ yields the quantization error

$$\mathbf{x} \;\; mod \;\; \Lambda = \mathbf{x} - \mathbf{Q}_{\Lambda}(\mathbf{x}). \tag{5}$$

The modulo operation always returns a point in the fundamental Voronoi region $\mathbb{V}_{\Lambda}$.

Definition 5 (Nested Lattice Codes): If a lattice Λ is a subset of another lattice Λ_{Fine}, $\Lambda \subset \Lambda_{Fine}$, then Λ is said to be nested in Λ_{Fine}. A nested lattice code $\mathscr{L}$ is the set of all points of a fine lattice Λ_{Fine} that are within the fundamental Voronoi region $\mathbb{V}_{\Lambda}$ of a coarse lattice Λ,

$$\mathscr{L} = \Lambda_{Fine} \cap \mathbb{V}_{\Lambda} = \left\{ \mathbf{t} : \mathbf{t} = \lambda \;\; mod \;\; \Lambda, \lambda \in \Lambda_{Fine} \right\}. \tag{6}$$



2. System Model

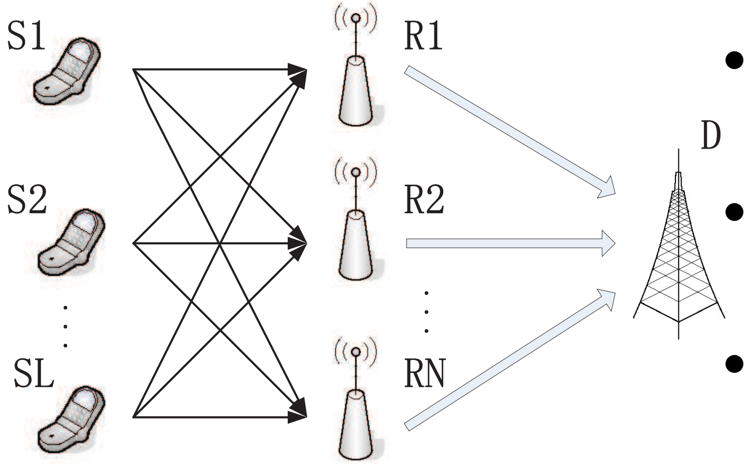


Figure 1: System model of a MSMR network

- L sources are communicating to one destination through N multiple relays.
- The first phase is for the transmissions from all sources to the relays.
- In the second phase, the relays compute-and-forward linear combinations of original messages towards the destination one by one.

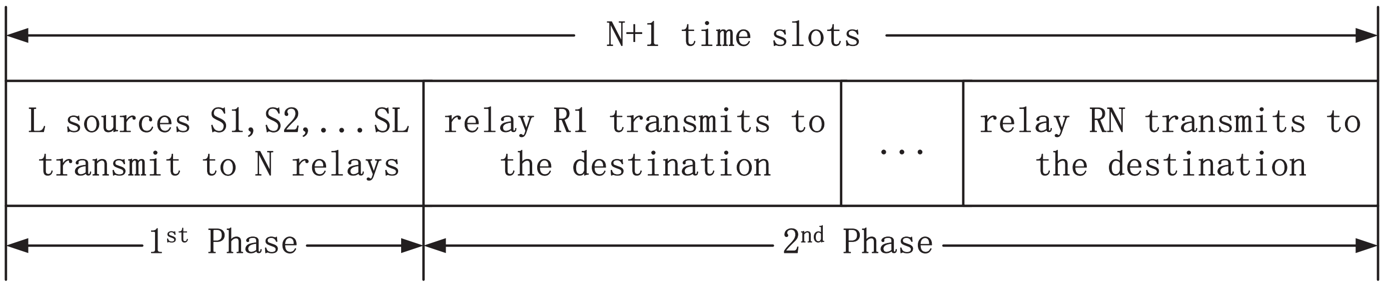


Figure 2: Time division allocation for one transmission realization

- Each source has a length-k vector over a finite field $\mathbb{F}_p = \{0, 1, \cdots, p-1\}$,

$$\mathbf{w}_l \in \mathbb{F}_p^k, \quad l = 1, 2, \cdots, L, \tag{7}$$

Encoder $\Psi_l : \mathbb{F}_p^k \rightarrow \mathbb{R}^n$ maps the length-k message $\mathbf{w}_l$ into a length-n lattice codeword $\mathbf{x}_l \in \mathbb{R}^n$, which satisfies the power constraint,

$$\frac{1}{n}||\mathbf{x}_l||^2 \leq P, \tag{8}$$

for $P \geq 0$ and $l = 1, 2, \cdots, L$.

- The m-th relay observes a noisy linear combination of the transmitted signals through the channel at the end of the first phase,

$$\mathbf{y}_m = \sum_{l=1}^{L} h_{ml}\mathbf{x}_l + \mathbf{z}_m, \qquad m = 1, 2, \cdots, L, \tag{9}$$

where $h_{ml} \in \mathbb{R}$ denotes real valued fading channel from $\mathcal{S}_l$ to the relay $\mathcal{R}_m$, generated i.i.d. according to $\mathcal{N}(0, 1)$; $\mathbf{z}_m \in \mathbb{R}^n$ denotes additive i.i.d. Gaussian noise vector, $\mathbf{z}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$. Let

$$\mathbf{h}_m = [h_{m1}, \cdots, h_{mL}]^T \tag{10}$$

denote the vector of channel coefficients from sources to the m-th relay.

3. Compute-and-Forward Scheme

• In recent work[a], Nazer and Gastpar propose the *compute-and-forward* approach that the relay nodes exploit the property that any integer combination of lattice points is again a lattice point.

• The m-th relay selects a scalar β_m and an integer network coding vector $\mathbf{a}_m = [a_{m1}, a_{m2}, \cdots, a_{mL}]^T$, $\mathbf{a}_m \in \mathbb{Z}^L$, and attempts to decode the lattice point $\sum_{l=1}^{L} a_{ml}\mathbf{x}_l$ from

$$\beta_m \mathbf{y}_m = \sum_{l=1}^{L} \beta_m h_{ml} \mathbf{x}_l + \beta_m \mathbf{z}_m \tag{11}$$

$$= \sum_{l=1}^{L} a_{ml}\mathbf{x}_l + \underbrace{\sum_{l=1}^{L} (\beta_m h_{ml} - a_{ml})\,\mathbf{x}_l + \beta_m \mathbf{z}_m}_{\text{Effective Noise}} \tag{12}$$

• In the finite field, it is equivalent that each relay is desired to reliable recover a linear combination of the messages,

$$\mathbf{u}_m = \bigoplus_{l=1}^{L} q_{ml}\mathbf{w}_l = \left[\sum_{l=1}^{L} a_{ml}\mathbf{w}_l\right] mod\ p, \tag{13}$$

where $\bigoplus$ denotes summation over the finite field, q_{ml} are coefficients taking values in $\mathbb{F}_p$.

[a] B. Nazer and M. Gastpar, "Compute-and-forward: harnessing interference through structured codes", *IEEE Trans. Info. Theory*, vol. 57, no. 10, pp. 6463-6486, Oct. 2011.

- Each relay is equipped with a decoder, $\Pi_m : \mathbb{R}^n \to \mathbb{F}_p^k$, that maps the observed channel output $\mathbf{y}_m \in \mathbb{R}^n$ to an estimate $\hat{\mathbf{u}}_m = \Pi_m(\mathbf{y}_m) \in \mathbb{F}_p^k$ of the message equation $\mathbf{u}_m$.

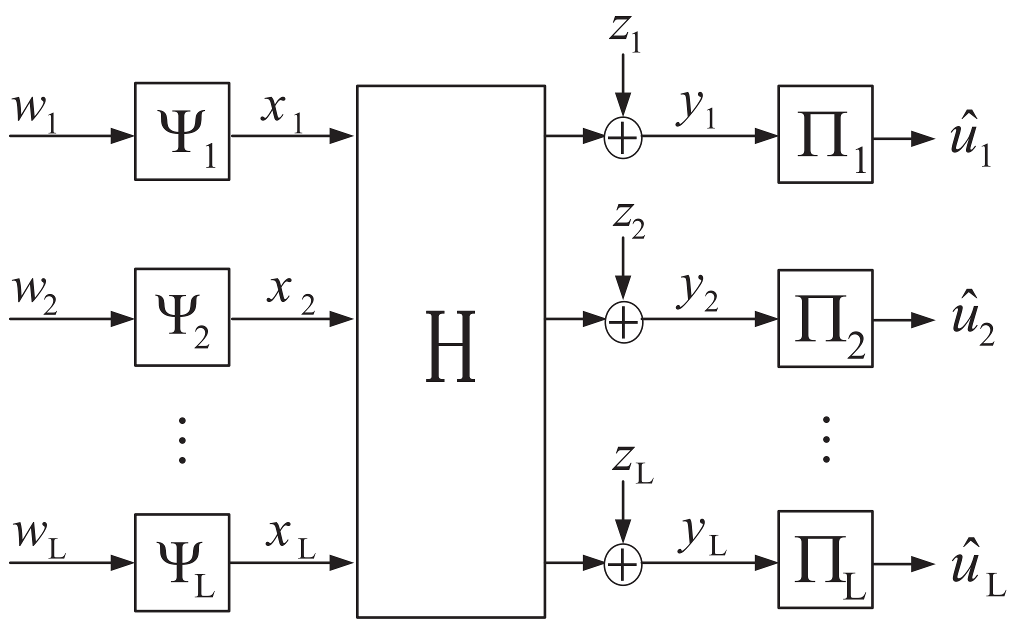


Figure 3: Compute-and-Forward Diagram

Theorem 1 : For real-valued AWGN networks with channel coefficient vector $\mathbf{h}_m \in \mathbb{R}^L$ and desired network coding vector $\mathbf{a}_m \in \mathbb{Z}^L$, the following computation rate is achievable

$$\mathscr{R}_m(\mathbf{a}_m) = \max_{\beta_m \in \mathbb{R}} \frac{1}{2} \log^+ \left(\frac{P}{\beta_m^2 + P||\beta_m \mathbf{h}_m - \mathbf{a}_m||^2} \right). \tag{14}$$

Theorem 2 : The computation rate given in Theorem 1 is uniquely maximized by choosing β_m to be the MMSE coefficient

$$\beta_{MMSE} = \frac{P\,\mathbf{h}_m^T \mathbf{a}_m}{1 + P||\mathbf{h}_m||^2}, \tag{15}$$

which results in a computation rate of

$$\mathscr{R}_m(\mathbf{a}_m) = \frac{1}{2} \log^+ \left(||\mathbf{a}_m||^2 - \frac{P(\mathbf{h}_m^T \mathbf{a}_m)^2}{1 + P||\mathbf{h}_m||^2} \right)^{-1}. \tag{16}$$

Theorem 3 : For a given channel coefficient vector $\mathbf{h}_m = [h_{m1}, h_{m2}, \cdots, h_{mL}]^T \in \mathbb{R}^L$, $\mathscr{R}_m(\mathbf{a}_m)$ is maximized by choosing the integer network coding vector $\mathbf{a}_m \in \mathbb{Z}^L$ as

$$\mathbf{a}_m = \textit{arg} \min_{\mathbf{a}_m \in \mathbb{Z}^L, \mathbf{a}_m \neq \mathbf{0}} \left(\mathbf{a}_m^T \mathbf{G}_m \mathbf{a}_m \right), \tag{17}$$

where

$$\mathbf{G}_m \triangleq \mathbf{I} - \frac{P}{1 + P\,||\mathbf{h}_m||^2} \mathbf{H}_m, \tag{18}$$

and $\mathbf{H}_m = [H_{ij}^{(m)}]$, $H_{ij}^{(m)} = h_{mi} h_{mj}$, $1 \le i, j \le L$.

4. Problem Statement

• Theorems 1-3 only give the optimal integer network coding vector $\mathbf{a}_m$ and achievable computation rate $\mathscr{R}_m$ for one relay and do not take consideration of the overall system constraints.

• The network coding system matrix $\mathbf{A}$ at the destination can be denoted as

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1L} \\ a_{21} & a_{22} & \cdots & a_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ a_{L1} & a_{L2} & \cdots & a_{LL} \end{bmatrix}. \tag{19}$$

The destination can solve for the original packets if the network coding system matrix $\mathbf{A}$ has full rank L, i.e. $|\mathbf{A}| \neq 0$. The message rates at the destination will be

$$\mathscr{R}_D = \min \{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_L\}. \tag{20}$$

• Instead of distributed calculations, we need to optimize the network coding vectors for L relays in a overall system level, by constructing the full rank network coding matrix $\mathbf{A}$ that makes the destination has maximum message rate.

- The optimization problem can be described as

$$\begin{aligned}\mathscr{R}_D^{max} &= \textit{arg} \max_{|\mathbf{A}|\neq 0} \left(\min\left\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_L\right\}\right) \\ &= \textit{arg} \max_{|\mathbf{A}|\neq 0} \left(\min_{m=1,\cdots L} \textit{arg} \max_{\mathbf{a}_m\in\mathbb{Z}^L, \mathbf{a}_m\neq \mathbf{0}} \frac{1}{2}\log^+ \left(||\mathbf{a}_m||^2 - \frac{P(\mathbf{h}_m^T\mathbf{a}_m)^2}{1+P||\mathbf{h}_m||^2} \right)^{-1} \right).\end{aligned}$$

In other words, we need to find the integer network coding vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_L$, under the system level constraint of $\mathbf{A}$ have full rank, such that the minimum value of $\mathscr{R}_1$, $\mathscr{R}_2$, $\cdots$, $\mathscr{R}_L$ is maximized.

- Equivalently, the optimum network coding integer coefficient matrix $\mathbf{A}$ should be

$$\mathbf{A} = \textit{arg} \min_{|\mathbf{A}|\neq 0} \left(\max_{m=1,\cdots L} \textit{arg} \min_{\mathbf{a}_m\in\mathbb{Z}^L, \mathbf{a}_m\neq \mathbf{0}} \left(\mathbf{a}_m^T \mathbf{G}_m \mathbf{a}_m\right)\right). \tag{21}$$

Proposed Strategy

Proposed Strategy:

• In the first step, for relay m, instead of finding one optimal network coding vector $\mathbf{a}_m$ to maximize its own computation rate, we are trying to find a candidate set,

$$\Omega_m^{T_{max}} = \{\mathbf{a}_m^{(1)}, \mathbf{a}_m^{(2)}, \cdots, \mathbf{a}_m^{(T_{max})}\}, \tag{22}$$

with $|\Omega_m^{T_{max}}| = T_{max}$. The network coding vectors with the top T_{max} maximum computation rates for relay m are within the candidate set $\Omega_m^{T_{max}}$.

FP Based Candidate Set $\Omega_m^{T_{max}}$ Searching Algorithm 1 is proposed.

• After we get all the candidate vector sets $\Omega_1^{T_{max}}$, $\Omega_2^{T_{max}}$, $\cdots$, $\Omega_L^{T_{max}}$, in the second step, we will try to pick up $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$, to construct the full rank network coding system matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T$, while in the meantime, the minimum corresponding $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ is maximized.

Network Coding System Matrix $\mathbf{A}$ Constructing Algorithm 2 is proposed.

1. Searching Candidate Set $\Omega_m^{T_{max}}$ for One Relay

- According to Theorem 3, it is equivalent to find the set $\Omega_m^{T_{max}}$ with length T_{max}, such that those vectors in $\Omega_m^{T_{max}}$ give the bottom T_{max} minimum $f(\mathbf{a}_m) \triangleq \mathbf{a}_m^T \mathbf{G}_m \mathbf{a}_m$ values.

- The searching of candidate set Ω_m^{max} with fixed length T_{max} can be decomposed into following steps.

(1) Enumerate all vectors $\mathbf{t} \in \mathbb{Z}^L$ $(\mathbf{t} \neq \mathbf{0})$ in Ω_m, such that $f(\mathbf{t}) = \mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C$ for a given positive constant C, i.e.,

$$\Omega_m = \left\{\mathbf{t} : \ f(\mathbf{t}) = \mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C, \ \mathbf{t} \neq \mathbf{0}, \ \mathbf{t} \in \mathbb{Z}^L \right\}. \tag{23}$$

(2) Adjust the constant C to guarantee that $|\Omega_m| \geq T_{max}$.

(3) Sort all the vectors $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_{|\Omega_m|}$ in Ω_m in descending order corresponding to the computation rate value $\mathscr{R}_m$ in (16), such that

$$\mathscr{R}_m(\mathbf{t}_1) \geq \mathscr{R}_m(\mathbf{t}_2) \geq \cdots \geq \mathscr{R}_m(\mathbf{t}_{|\Omega_m|}). \tag{24}$$

(4) Pick the first T_{max} vectors of Ω_m to form the set $\Omega_m^{T_{max}}$.

- The procedure of enumerating all vectors $\mathbf{t} \in \mathbb{Z}^L$ ($\mathbf{t} \neq \mathbf{0}$) in Ω_m, such that $f(\mathbf{t}) = \mathbf{t}^T\mathbf{G}_m\mathbf{t} \leq C$ for a given positive constant C is based on the Fincke-Pohst Method.

- Operate Cholesky's factorization of matrix $\mathbf{G}_m$ which yields $\mathbf{G}_m = \mathbf{U}^T\mathbf{U}$, where $\mathbf{U}$ is an upper triangular matrix. Let u_{ij}, $i, j = 1, \cdots, L$, be the entries of matrix $\mathbf{U}$ and $\mathbf{t} = [t_1, t_2, \cdots, t_L]^T$. Then, the searching vector $\mathbf{t}$ that make $\mathbf{t}^T\mathbf{G}_m\mathbf{t} \leq C$ can be expressed as

$$\begin{aligned} \mathbf{t}^T\mathbf{G}_m\mathbf{t} &= ||\mathbf{U}\ \mathbf{t}||_F^2 = \sum_{i=1}^{L}\left(u_{ii}t_i + \sum_{j=i+1}^{L} u_{ij}t_j\right)^2 \\ &= \sum_{i=k}^{L} g_{ii}\left(t_i + \sum_{j=i+1}^{L} g_{ij}t_j\right)^2 + \sum_{i=1}^{k-1} g_{ii}\left(t_i + \sum_{j=i+1}^{L} g_{ij}t_j\right)^2 \leq C \end{aligned} \tag{25}$$

where $g_{ii} = u_{ii}^2$ and $g_{ij} = u_{ij}/u_{ii}$ for $i = 1, 2, \cdots, L$, $j = i+1, \cdots, L$.

- Obviously the second term of (25) is non-negative, hence, it is equivalent to consider for every $k = L, L-1, \cdots, 1$,

$$\sum_{i=k}^{L} g_{ii}\left(t_i + \sum_{j=i+1}^{L} g_{ij}t_j\right)^2 \leq C. \tag{26}$$

Then, we can start work backwards to find the bounds for vector entries $t_L, t_{L-1}, \cdots, t_1$ one by one.

Algorithm 1 FP Based Candidate Set $\Omega_m^{T_{max}}$ Searching Algorithm

Input: Matrix $\mathbf{G}_m$, $T_{max} = |\Omega_m^{T_{max}}|$.

Output: The candidate set $\Omega_m^{T_{max}}$ and corresponding computation rate set $\Gamma_m^{T_{max}}$.

Step 1: Calculate the binary quantized vector obtained by applying the direct sign operator of the real minimum-eigenvalue eigenvector of $\mathbf{G}_m$, denoted as $\mathbf{t}_{quant}$, and set C as

$$C = \mathbf{t}_{quant}^T \mathbf{G}_m \, \mathbf{t}_{quant}. \tag{27}$$

Step 2: Operate Cholesky's factorization of matrix $\mathbf{G}_m$ yields $\mathbf{G}_m = \mathbf{U}^T\mathbf{U}$, where $\mathbf{U}$ is an upper triangular matrix. Let u_{ij}, $i, j = 1, \cdots, L$, denote the entries of matrix $\mathbf{U}$. Set

$$g_{ii} = u_{ii}^2, \qquad g_{ij} = u_{ij}/u_{ii}, \qquad i = 1, \cdots, L, j = i + 1, \cdots, L.$$

Step 3: Search set $\Omega_m = \left\{\mathbf{t} : \ \mathbf{t}^T\mathbf{G}_m\mathbf{t} \le C, \ \mathbf{t} \ne \mathbf{0}, \ \mathbf{t} \in \mathbb{Z}^L\right\}$ according to the following Fincke-Pohst procedure.

(i) Start from $\Delta_L = 0$, $C_L = C$, $k = L$ and $\Omega_m = \emptyset$.

(ii) Set the upper bound UB_k and the lower bound LB_k as follows

$$UB_k = \left\lfloor \sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rfloor, \qquad LB_k = \left\lceil -\sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rceil,$$

and $t_k = LB_k - 1$.

(iii) Set $t_k = t_k + 1$. For $t_k \leq UB_k$, go to (v); else go to (iv).

(iv) If $k = L$, terminate and output Ω_m; else set $k = k + 1$ and go to (iii).

(v) For $k = 1$, go to (vi); else set $k = k - 1$, and

$$\Delta_k = \sum_{j=k+1}^{L} g_{kj} t_j, \qquad C_k = C_{k+1} - g_{k+1,k+1} \left(\Delta_{k+1} + t_{k+1}\right)^2,$$

then go to (ii).

(vi) If $\mathbf{t} = \mathbf{0}$ terminate, else we get a candidate vector $\mathbf{t} \neq \mathbf{0}$ that satisfies all the bounds requirements and put it inside Ω_m, i.e., $\Omega_m = \{\Omega_m, \mathbf{t}\}$. Go to (iii).

Step 4: If $|\Omega_m| < T_{max}$, set $C = 2C$ and repeat Step 3.

Step 5: Sort all the vectors $\mathbf{t}_1$, $\mathbf{t}_2$, $\cdots$, $\mathbf{t}_{|\Omega_m|}$ in Ω_m in descending order corresponding to the computation rate value $\mathscr{R}_m$, such that $\mathscr{R}_m(\mathbf{t}_1) \geq \mathscr{R}_m(\mathbf{t}_2) \geq \cdots \geq \mathscr{R}_m(\mathbf{t}_{|\Omega_m|})$.

Pick the first T_{max} vectors of Ω_m to form the set $\Omega_m^{T_{max}}$ and construct the corresponding computation rate $\Gamma_m^{T_{max}}$ as

$$\begin{cases} \Omega_m^{T_{max}} = \{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_{T_{max}}\}, \\ \Gamma_m^{T_{max}} = \{\mathscr{R}_m(\mathbf{t}_1), \mathscr{R}_m(\mathbf{t}_2), \cdots, \mathscr{R}_m(\mathbf{t}_{T_{max}})\}. \end{cases} \tag{28}$$

2. Constructing Network Coding Matrix A

- After running our proposed FP Based Candidate Set $\Omega_m^{T_{max}}$ Searching Algorithm for each relay, we can have two length-T_{max} tables.

$$\textit{Table 1:} \qquad \Gamma_m^{T_{max}} = \{\mathscr{R}_m^{(1)}, \mathscr{R}_m^{(2)}, \cdots, \mathscr{R}_m^{(T_{max})}\}, \tag{29}$$

$$\textit{Table 2:} \qquad \Omega_m^{T_{max}} = \{\mathbf{a}_m^{(1)}, \mathbf{a}_m^{(2)}, \cdots, \mathbf{a}_m^{(T_{max})}\}. \tag{30}$$

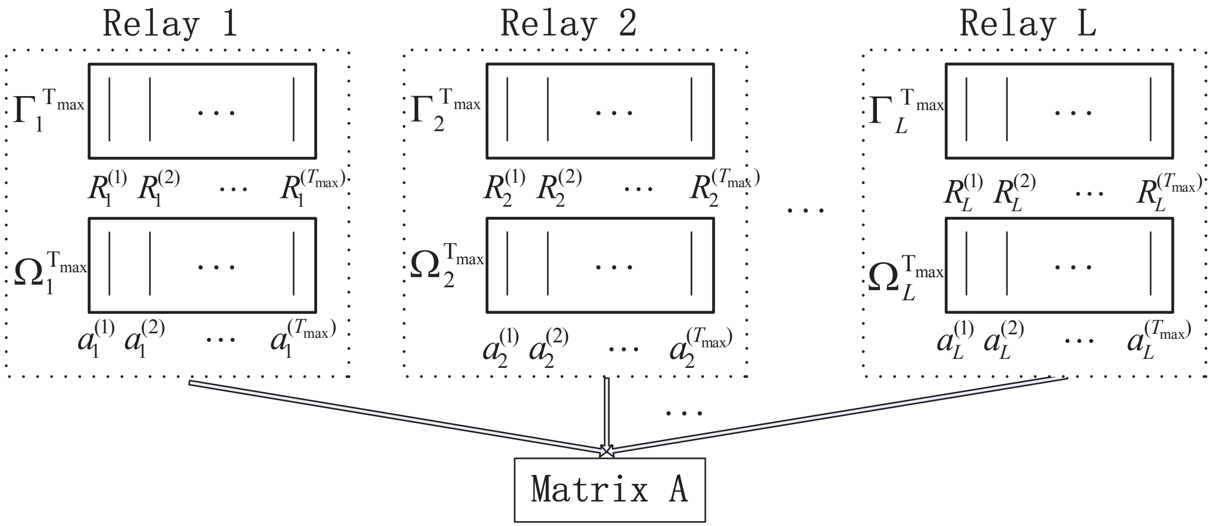


Figure 4: Candidate sets and rate tables for all relays

- We will try to pick up $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$, to construct the system network coding matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T$ with full rank, at the same time, the minimum corresponding rate $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ is maximized.

• First, we sort the overall computation rate set for all relays $\{\Gamma_1^{T_{max}}, \Gamma_2^{T_{max}}, \cdots, \Gamma_L^{T_{max}}\}$ in a descending order into

$$\{\gamma_1, \gamma_2, \cdots, \gamma_{L\times T_{max}}\}: \quad \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{L\times T_{max}}. \tag{31}$$

• Then, start from the largest possible achievable rate γ_{index} with $index = L$ (the first $L-1$ rates are obviously not achievable), we will check whether the rate is achievable, which means we can find L vectors $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$, such that two constraints need to be satisfied:

(i) The system network coding matrix $\mathbf{A}$ is full rank;

(ii) $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ all greater or equal to γ_{index}.

If we cannot find vectors satisfy those constraints, we move to the next largest possible achievable rate $\gamma_{index+1}$ and check in the same way, and so on, until the first achievable rate is found.

• When we are checking one possible achievable rate γ_{index}, we will reduce/cut the network coding candidate sets $\Omega_1^{T_{max}}, \cdots, \Omega_L^{T_{max}} \quad \rightarrow \quad \Omega_1^{cut}, \cdots, \Omega_L^{cut}$, in which any $\mathbf{a}_1 \in \Omega_1^{cut}$, any $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, any $\mathbf{a}_L \in \Omega_L^{cut}$ will satisfy that $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ all greater or equal to γ_{index}.

- Suppose $\gamma_{index} = \mathscr{R}_m^{(n)} \in \Gamma_m^{T_{max}}$, i.e., γ_{index} is taken from Table 1 of relay m with table index n, then the network coding vector $\mathbf{a}_m$ is taken from Table 2 with same index n, i.e., $\mathbf{a}_m = \mathbf{a}_m^{(n)} \in \Omega_m^{max}$ is fixed for that relay and $\Omega_m^{cut} = \{\mathbf{a}_m\}$.
- For other relays $i \neq m$, the candidate set will reduce/cut to length T_i^{cut} such that $\mathscr{R}_i^{(1)}, \mathscr{R}_i^{(2)}, \cdots, \mathscr{R}_i^{(T_i^{cut})}$ all greater or equal to γ_{index}. Set $\Omega_i^{cut} = \{\mathbf{a}_i^{(1)}, \mathbf{a}_i^{(2)}, \cdots, \mathbf{a}_i^{(T_i^{cut})}\}$.
- We can check the constraint *(i)* about the system network coding matrix $\mathbf{A}$ constructed by $\mathbf{a}_1 \in \Omega_1^{cut}, \cdots, \mathbf{a}_L \in \Omega_L^{cut}$, if it is of full rank, then this rate γ_{index} is achievable.

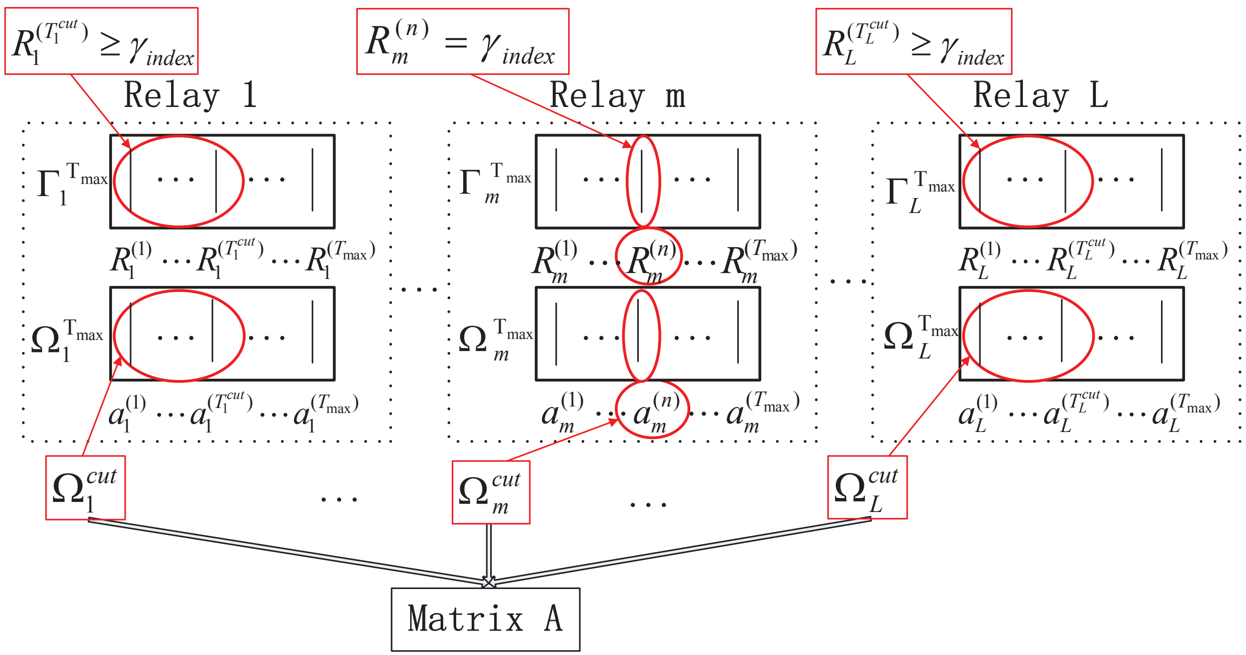


Figure 5: Constructing network coding system matrix $\mathbf{A}$

Algorithm 2 Network Coding System Matrix $\mathbf{A}$ Constructing Algorithm

Input: Candidate sets $\Omega_1^{T_{max}}, \cdots, \Omega_L^{T_{max}}$; Computation rate sets $\Gamma_1^{T_{max}}, \cdots, \Gamma_L^{T_{max}}$.

Output: The network coding system matrix $\mathbf{A}$ constructed from $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$ with full rank that gives the maximum achievable rate $\mathscr{R}_D^{max}$.

Step 1: Sort the overall computation rate set for all relays $\{\Gamma_1^{T_{max}}, \Gamma_2^{T_{max}}, \cdots, \Gamma_L^{T_{max}}\}$ in a descending order into $\{\gamma_1, \gamma_2, \cdots, \gamma_{L\times T_{max}}\}$, such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{L\times T_{max}}$. Initialize *index* $= L$.

Step 2: Check whether the rate of γ_{index} is achievable by the following procedure. Suppose $\gamma_{index} = \mathscr{R}_m^{(n)} \in \Gamma_m^{T_{max}}$. Then, for relay i, the reduced candidate set $\Omega_i^{cut}, i = 1, 2, \cdots, L$ will be constructed as follows.

(i) For relay m, set $\Omega_m^{cut} = \{\mathbf{a}_m^{(n)}\}$.

(ii) For relay $i \neq m$, compare the value of γ_{index} and the sorted descending set $\Gamma_i^{T_{max}} = \{\mathscr{R}_i^{(1)}, \mathscr{R}_i^{(2)}, \cdots, \mathscr{R}_i^{(T_{max})}\}$, and find all $\{\mathscr{R}_i^{(1)}, \mathscr{R}_i^{(2)}, \cdots, \mathscr{R}_i^{(T_i^{cut})}\}$ greater or equal to γ_{index}. Set $\Omega_i^{cut} = \{\mathbf{a}_i^{(1)}, \mathbf{a}_i^{(2)}, \cdots, \mathbf{a}_i^{(T_i^{cut})}\}$.

Step 3: Check every $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{cut}$, until we find one network coding system matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T$ has full rank, i.e., $|\mathbf{A}| \neq 0$. If so, terminate and output the network coding system matrix $\mathbf{A}$ and the maximum achievable rate $\mathscr{R}_D^{max} = \gamma_{index}$.

Step 4: If for any $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{cut}$, we cannot construct a full rank network coding system matrix $\mathbf{A}$, then set *index* $=$ *index* $+ 1$, go to Step 2.

A Transparent Realization

- For a three sources three relays system, $L = N = 3$, the power constraints $P = 10dB$ and $T_{max} = 5$. The channel coefficient vector $\mathbf{h}_m$ for each relay is

$$\mathbf{h}_1 = [0.9730, 0.4674, 0.5103]^T, \quad \mathbf{h}_2 = [-1.7291, 0.7166, -0.5856]^T, \quad \mathbf{h}_3 = [-0.3912, 1.4407, -0.8115]^T.$$

- Then we can calculated $\mathbf{G}_m$ by Theorem 3,

$$\mathbf{G}_1 = \begin{bmatrix} 0.3794 & -0.2981 & -0.3254 \\ -0.2981 & 0.8568 & -0.1563 \\ -0.3254 & -0.1563 & 0.8293 \end{bmatrix}, \mathbf{G}_2 = \begin{bmatrix} 0.2424 & 0.3140 & -0.2566 \\ 0.3140 & 0.8699 & 0.1063 \\ -0.2566 & 0.1063 & 0.9131 \end{bmatrix}, \mathbf{G}_3 = \begin{bmatrix} 0.9488 & 0.1887 & -0.1063 \\ 0.1887 & 0.3052 & 0.3914 \\ -0.1063 & 0.3914 & 0.7796 \end{bmatrix}.$$

- We can see that if we optimize the network coding vectors separately,

$$\begin{aligned} \mathbf{a}_1^{opt} &= \mathit{arg} \min_{\mathbf{a}_1 \in \mathbb{Z}^L, \mathbf{a}_1 \neq \mathbf{0}} \left(\mathbf{a}_1^T \mathbf{G}_1 \mathbf{a}_1\right) = [1\ 0\ 0]^T, \\ \mathbf{a}_2^{opt} &= \mathit{arg} \min_{\mathbf{a}_2 \in \mathbb{Z}^L, \mathbf{a}_2 \neq \mathbf{0}} \left(\mathbf{a}_2^T \mathbf{G}_2 \mathbf{a}_2\right) = [1\ 0\ 0]^T, \\ \mathbf{a}_3^{opt} &= \mathit{arg} \min_{\mathbf{a}_3 \in \mathbb{Z}^L, \mathbf{a}_3 \neq \mathbf{0}} \left(\mathbf{a}_3^T \mathbf{G}_3 \mathbf{a}_3\right) = [0\ -1\ 1]^T. \end{aligned}$$

- The corresponding network coding system matrix $\mathbf{A}_{separate}$ is

$$\mathbf{A}_{separate} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Obviously, optimizing the network coding coefficient vectors separately cannot satisfy the system constraints since the matrix $\mathbf{A}_{separate}$ is not full rank.

- After running our proposed FP Based Candidate Set $\Omega_m^{T_{max}}$ Searching Algorithm 1 for each relay,

$$\Omega_1^{T_{max}} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad \Gamma_1^{T_{max}} = [0.4846, \ 0.4620, \ 0.3408, \ 0.2918, \ 0.2231];$$

$$\Omega_2^{T_{max}} = \begin{bmatrix} 1 & 2 & 3 & -1 & -2 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \Gamma_2^{T_{max}} = [0.7087, \ 0.6785, \ 0.5572, \ 0.3625, \ 0.2694];$$

$$\Omega_3^{T_{max}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & -2 & -2 & -3 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}, \quad \Gamma_3^{T_{max}} = [0.5987, \ 0.5935, \ 0.4384, \ 0.4165, \ 0.2902].$$

- Now we are running our proposed Network Coding System Matrix $\mathbf{A}$ Constructing Algorithm 2. First we sort the overall computation rate set for all relays $\{\Gamma_1^{T_{max}}, \Gamma_2^{T_{max}}, \Gamma_3^{T_{max}}\}$ in a descending order into

$$\{\underbrace{0.7087}_{\gamma_1}, \underbrace{0.6785}_{\gamma_2}, \underbrace{0.5987}_{\gamma_3}, \underbrace{0.5935}_{\gamma_4}, \underbrace{0.5572}_{\gamma_5}, \underbrace{0.4846}_{\gamma_6}, \cdots\}. \tag{32}$$

We will start to check the rate from the third maximum, $\gamma_3 = 0.5987$, then $\gamma_4 = 0.5935$, then $\gamma_5 = 0.5572$, $\cdots$, to see whether it is achievable. If so, terminate and output; if not, move to the next maximum rate.

- For example, when we are checking $\gamma_4 = 0.5935 = \mathscr{R}_3^{(2)}$, the reduced candidate sets Ω_1^{cut}, Ω_2^{cut}, Ω_3^{cut} with all corresponding rate greater or equal to $\gamma_4 = 0.5935$ can be constructed as

$$\Omega_1^{cut} = \emptyset, \qquad \Omega_2^{cut} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \Omega_3^{cut} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{33}$$

We can easily see no full rank network coding system matrix $\mathbf{A}$ can be constructed with $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\mathbf{a}_3 \in \Omega_3^{cut}$, the rate of $\gamma_4 = 0.5935$ is not achievable. We will move to $\gamma_5 = 0.5572$ and check in the same way.

- After running our proposed Network Coding System Matrix $\mathbf{A}$ Constructing Algorithm 2, the network coding system matrix $\mathbf{A}$ is finally constructed as

$$\mathbf{A}_{proposed} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{34}$$

and the maximum achievable rate $\mathscr{R}_D^{max} = 0.4846$.

Experimental Studies

In Fig. 6, we show that if network coding integer vector is optimized separately/locally at each relay, the probability that the network coding system matrix $\mathbf{A}$ is not of full rank, i.e. $|\mathbf{A}| = 0$, in which case the destination actually cannot decode the original messages efficiently.

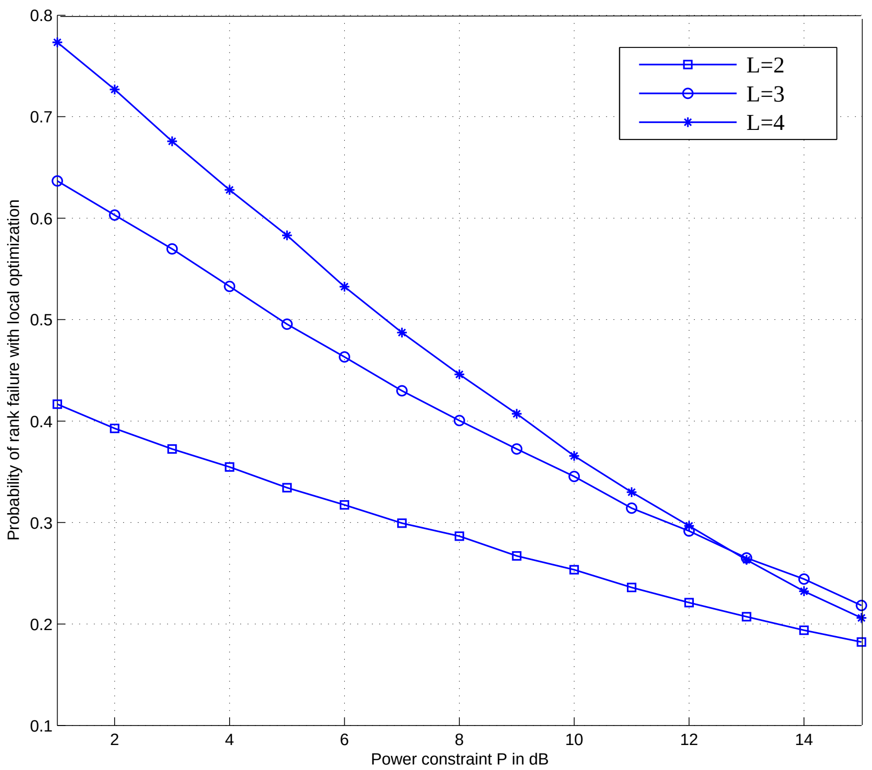


Figure 6: Probability of rank failure with local optimization for $L = 2, 3, 4$

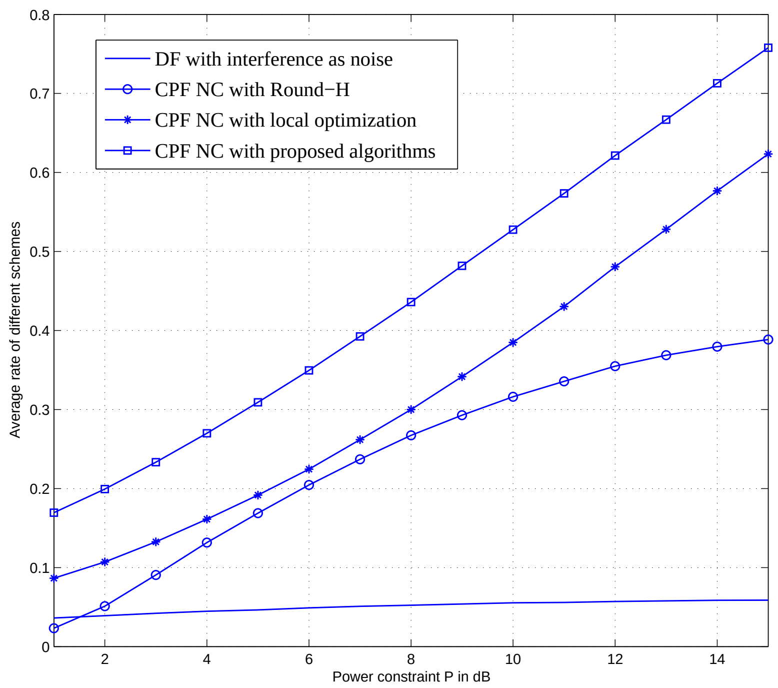


Figure 7: Rate comparisons for $L = 3$ and $T_{max} = 5$

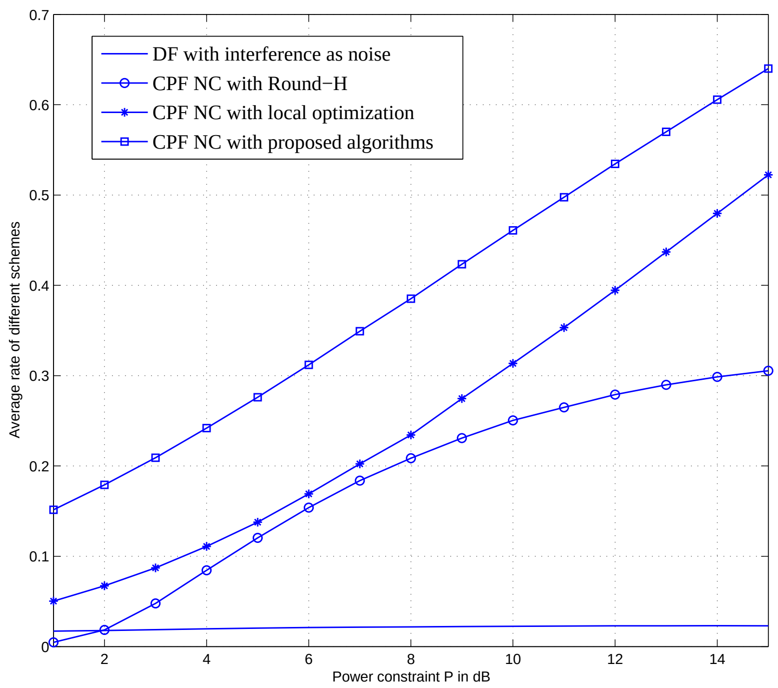


Figure 8: Rate comparisons for $L = 4$ and $T_{max} = 5$

Conclusions

- We consider integer network coding design in a system level over a compute-and-forward multi-source multi-relay system.
- We propose the Fincke-Pohst based candidate set searching algorithm, to provide a network coding vector candidate set for each relay with corresponding computation rate in descending order.
- Then, with our proposed network coding system matrix constructing algorithm, we choose network coding vectors from candidate sets to construct network coding system matrix with full rank, while in the meantime the transmission rate of the overall system is maximized.

References

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Thank You!