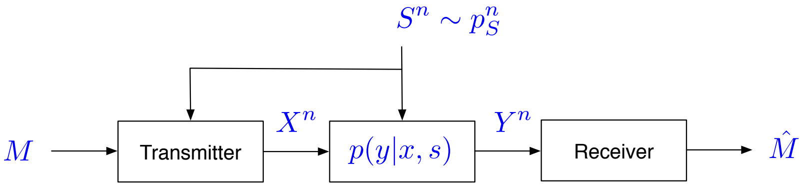
Approximately Optimal Distributions via the ADT Linear Deterministic Model

Tie Liu

June-19-2014 @CUHK

Communication with side information



- S^n: Non-causally known at the transmitter as side information

What is the capacity of the channel?

Channel capacity

- A simple upper bound:

$$C \leq \max_{p(x|s)} I(X;Y|S)$$

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  - $|\mathcal{U}| \leq \min\{|\mathcal{X}||\mathcal{S}|, |\mathcal{Y}| + |\mathcal{S}| - 1\}$

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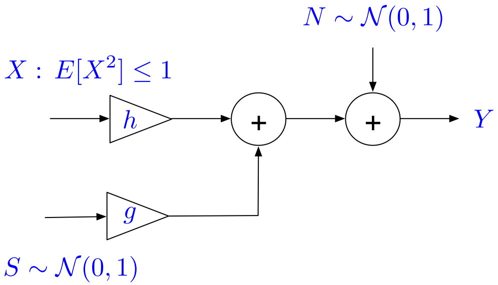
One can get "lucky" though ...

Writing on dirty paper



- $X \sim \mathcal{N}(0,1)$, $X \perp S$, and $U = hX + \frac{h^2}{h^2+1} gS$ (Costa 1983):

$$I(U;Y) - I(U;X) = \frac{1}{2}\log(1+h^2)$$

which coincides with the simple upper bound

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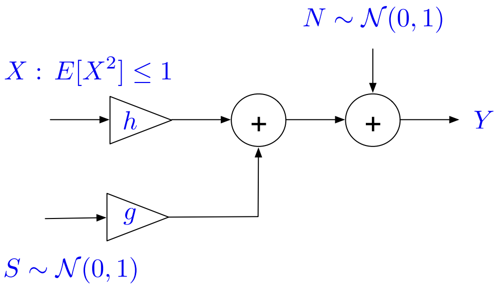

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However, "luck" may be running out sometimes ...

Running out of "luck"

- No obvious choice of input/auxiliary random variables in the single-letter capacity/achievable rate expressions to match the simple upper bound:
  - Writing on fading paper
  - Secret writing on dirty paper
  - Multiple-user writing on dirty paper

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What can we do?

This talk

- Goal: A systematic approach to identify approximately optimal choice of input/auxiliary random variables

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- Plan:
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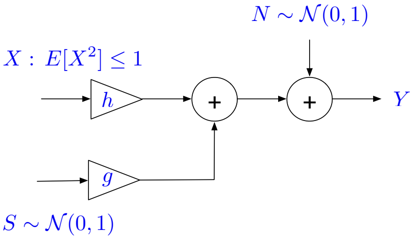
This talk

- Goal: A systematic approach to identify approximately optimal choice of input/auxiliary random variables
- Approach: To take a deterministic view (Avestimehr-Diggavi-Tse 2007)
- Plan:
  - Revisit Costa's dirty-paper channel
  - Apply the insight to the problems of: 1) secret writing on dirty paper; and 2) two-user symmetric Gaussian interference channel

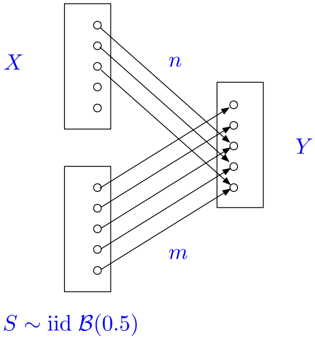
Writing on dirty paper

Gaussian model



$$Y = hX + gS + N$$

ADT linear deterministic model



$$Y = D_q^{q-n} X + D_q^{q-m} S$$

Capacity of deterministic model

- Y is a deterministic function of (X, S):
  - Simplifying the upper bound:

$$\begin{aligned} C &\leq \max_{p(x|s)} I(X;Y|S) \\ &= \max_{p(x|s)} H(Y|S) - H(Y|X,S) \\ &= \max_{p(x|s)} H(Y|S) \end{aligned}$$

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  - Choosing $U = Y$:

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  - Conclusion:

$$C = \max_{p(x|s)} H(Y|S)$$

Capacity of ADT linear deterministic model

- For ADT linear deterministic model:

$$
\begin{aligned}
H(Y|S) &= H(D_q^{q-n}X + D_q^{q-m}S|S) \\
&\leq H(D_q^{q-n}X) \\
&\leq \mathrm{rank}(D_q^{q-n}) \\
&= n
\end{aligned}
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where equality holds when X is Bernoulli-$1/2$ and independent of S

Translation to Gaussian model

- ADT linear deterministic model (an optimal choice):

$$U = Y = D_q^{q-n} X + D_q^{q-m} S$$

where X is i.i.d. Bernoulli-$1/2$ and independent of S

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How good is this choice of (U, X)?

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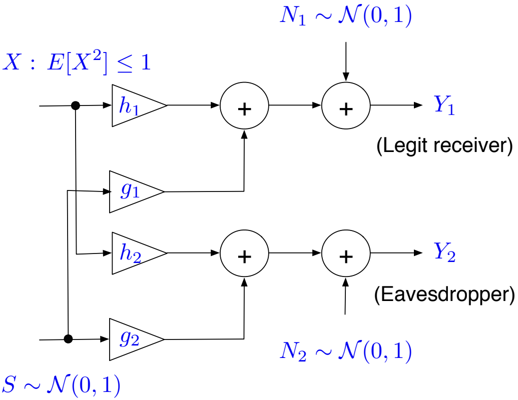
How robust is this approach?

Secret writing on dirty paper



- S^n: Non-causally known at the transmitter as side information
- Secrecy constraint: $(1/t)I(M; Y_2^t) \to 0$

Secrecy capacity bounds

- A single-letter achievable secrecy rate (Chen-Vinck 2008):

$$C_s \geq \max_{p(u,x|s)} \min[I(U;Y_1) - I(U;S), I(U;Y_1) - I(U;Y_2)]$$

  - Achieved by a double binning scheme

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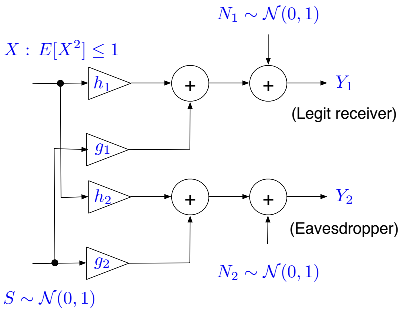
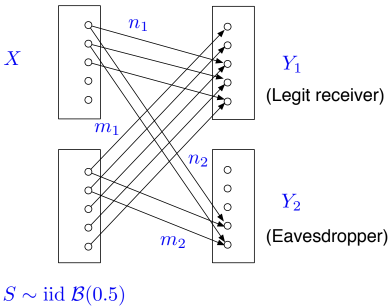
Let' try the deterministic approach ...

Secret writing on dirty paper

Gaussian model



ADT linear deterministic model



$$
\begin{aligned}
Y_1 &= h_1 X + g_1 S + N_1 \\
Y_2 &= h_2 X + g_2 S + N_2
\end{aligned}
$$

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\begin{aligned}
Y_1 &= D_q^{q-n_1} X + D_q^{q-m_1} S \\
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Secrecy capacity of semi-deterministic model

- Y_1 is a deterministic function of (X, S):
  - Simplifying the upper bound:

$$\begin{aligned} C_s &\leq \max_{p(x|s)} \min[I(X;Y_1|S), I(X,S;Y_1|Y_2)] \\ &= \max_{p(x|s)} \min[H(Y_1|S) - H(Y_1|X,S), H(Y_1|Y_2) - H(Y_1|X,S,Y_2)] \\ &= \max_{p(x|s)} \min[H(Y_1|S), H(Y_1|Y_2)] \end{aligned}$$

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  - Choosing $U = Y_1$:

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  - Conclusion:

$$C_s = \max_{p(x|s)} \min[H(Y_1|S), H(Y_1|Y_2)]$$

Secrecy capacity of ADT linear deterministic model

- For ADT linear deterministic model:
  - First:

$$\begin{aligned} H(Y_1|S) &= H(D_q^{q-n_1}X + D_q^{q-m_1}S|S) \\ &\leq H(D_q^{q-n_1}X) \\ &\leq \mathrm{rank}(D_q^{q-n_1}) \\ &= n_1 \end{aligned}$$

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  - Second:

$$\begin{aligned} H(Y_1|Y_2) &= H(D_q^{q-n_1}X + D_q^{q-m_1}S|D_q^{q-n_2}X + D_q^{q-m_2}S) \\ &= H\left(\left[D_q^{q-n_1} \; D_q^{q-m_1}\right] \begin{bmatrix} X \\ S \end{bmatrix} \middle| \left[D_q^{q-n_2} \; D_q^{q-m_2}\right] \begin{bmatrix} X \\ S \end{bmatrix} \right) \end{aligned}$$

A technical lemma

Let A and B be two matrices in $\mathbb{F}_2$ with the same number of columns. Then

$$\max H(AZ|BZ) = \text{rank}\left(\left[\begin{array}{c} A \\ B \end{array}\right]\right) - \text{rank}(B)$$

where the maximization is over all possible binary random vectors Z. The maximization is achieved when Z is i.i.d. Bernoulli-$1/2$

Translation to Gaussian model

- ADT linear deterministic model (an optimal choice):

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- The simple upper bound:

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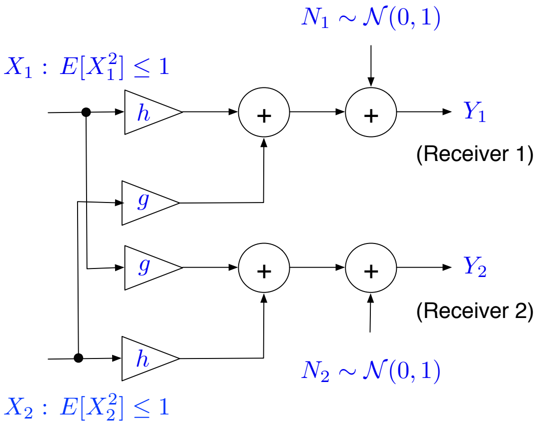
- Secrecy capacity to within $1/2$ bit

Mustafa El-Halabi, Tie Liu, Costas N. Georghiades, and Shlomo Shamai (Shitz), "Secret writing on dirty paper: A deterministic view," *IEEE Transactions on Information Theory*, vol. 58, no. 6, pp. 3419–3429, June 2012

Two-user symmetric Gaussian interference channel



- Two independent messages, one between each transmitter-receiver pair

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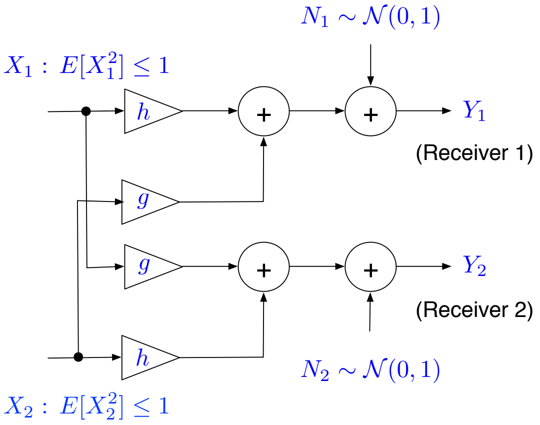

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What is the sum capacity of the channel?

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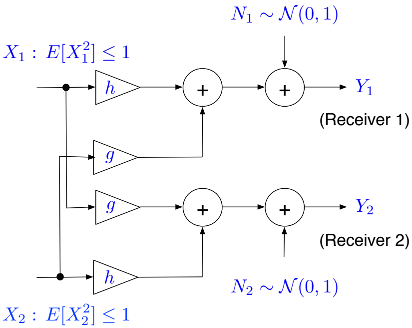
  - Sum capacity to within one bit (Etkin-Tse-Wang 2008)

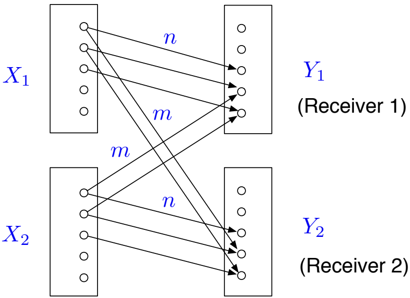
Two-user symmetric interference channel

Gaussian model



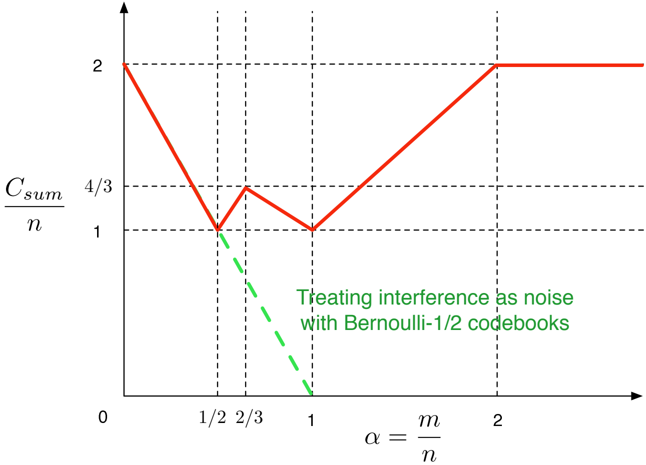
$$\begin{aligned} Y_1 &= hX_1 + gX_2 + N_1 \\ Y_2 &= gX_1 + hX_2 + N_2 \end{aligned}$$

ADT linear deterministic model



$$\begin{aligned} Y_1 &= D_q^{q-n} X_1 + D_q^{q-m} X_2 \\ Y_2 &= D_q^{q-m} X_1 + D_q^{q-n} X_2 \end{aligned}$$

Sum capacity of ADT linear deterministic model



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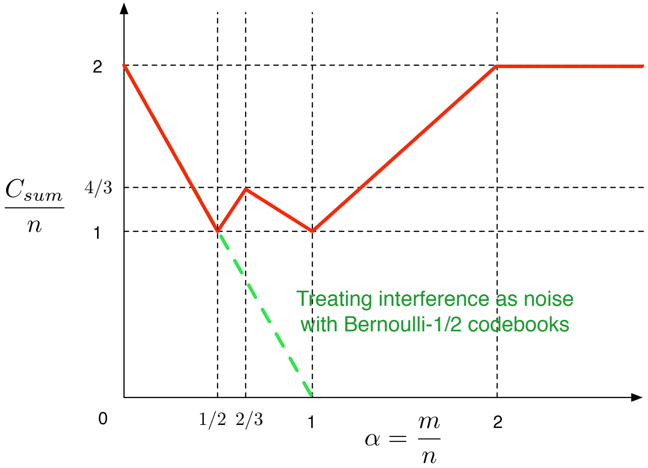

Can the simple strategy of treating interference as noise be good beyond the "very-weak" interference regime?

The limit of treating interference as noise

- Treating interference as noise can be arbitrarily good:

$$C_{sum} = \lim_{k\to\infty} \frac{C_{sum}^{(k)}}{k}$$

where

$$C_{sum}^{(k)} := \max_{p(x_1^k)p(x_2^k)} \left[I(X_1^k; Y_1^k) + I(X_2^k; Y_2^k) \right]$$

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Again let' try the deterministic approach ...

ADT linear deterministic channel

- Fix k:

$$I(X_1^k;Y_1^k) = H(AX_1^k + BX_2^k) - H(BX_2^k)$$
$$I(X_2^k;Y_2^k) = H(BX_1^k + AX_2^k) - H(BX_1^k)$$

where A and B are kth Kronecker power of D_q^{q-n} and D_q^{q-m}, respectively

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- Choose $X_1^k = EZ_1$ and $X_2^k = EZ_2$ where Z_1 and Z_2 are i.i.d. Bernoulli-1/2 vectors for some E of kq rows:

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  - $(1, I_q)$ is sufficient for the "very-weak" interference regime
  - What about the other regimes?

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- Clearly,

$$rank([AE\ BE]) - rank(BE) = 2n - n = n = \frac{C_{sum}}{2}$$

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- May require k up to 2

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- Sum capacity within $\log\log\max(|h|^2, |g|^2)$ bits (preliminary analysis)

Summary

- Identifying an optimal choice of input/auxiliary random variables in a single/multi-letter capacity/achievable rate expression for Gaussian networks can be extremely challenging

- We look for a more systematic search guided by the ADT linear deterministic model:
  - May settle for approximate optimality

- A more refined deterministic model (than the ADT linear deterministic model) might be needed to achieve universal approximation