A Graph Theoretical Approach to Network Encoding Complexity

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University of Hong Kong

February, 2012

Merging

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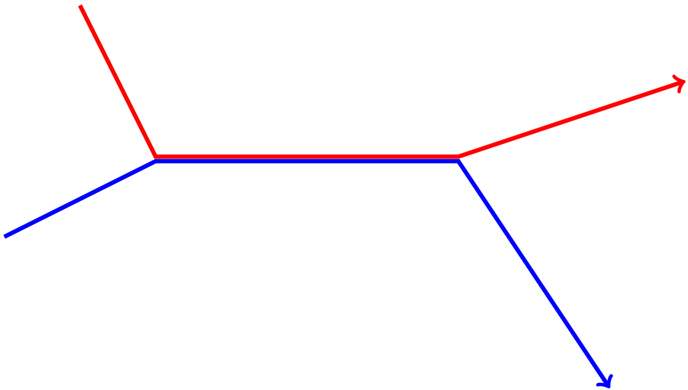

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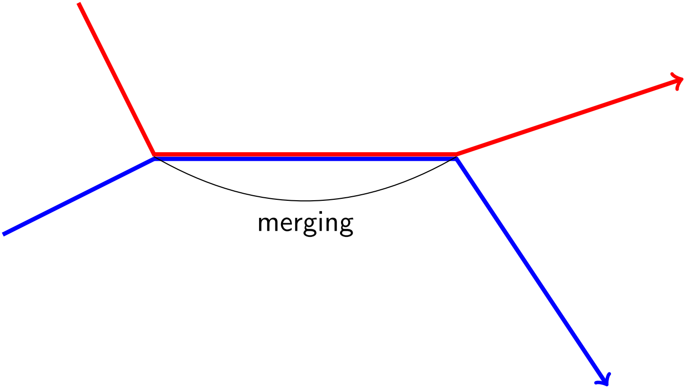

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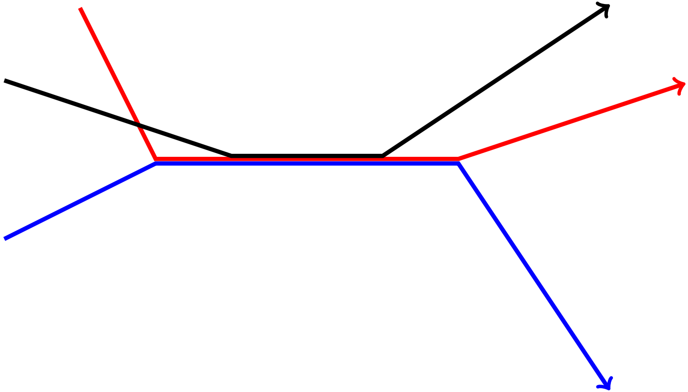

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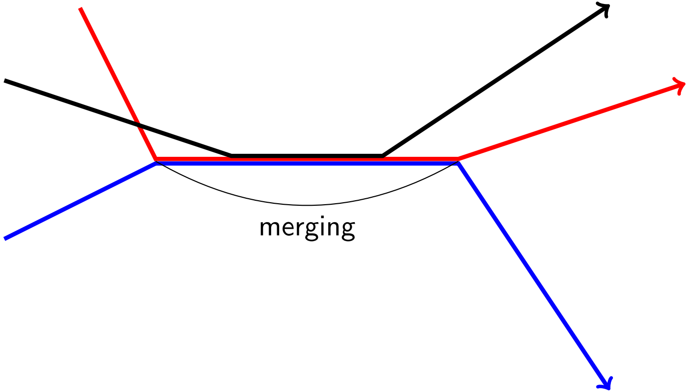

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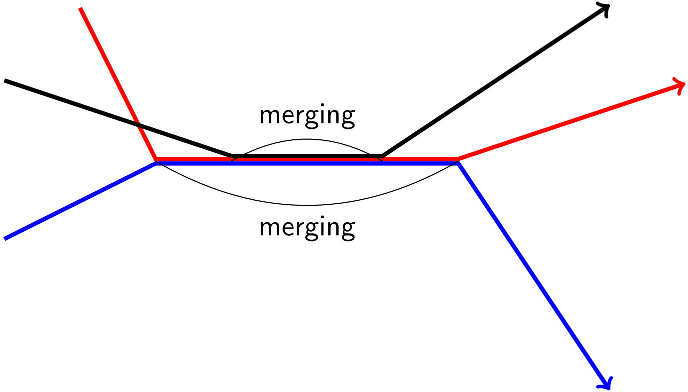
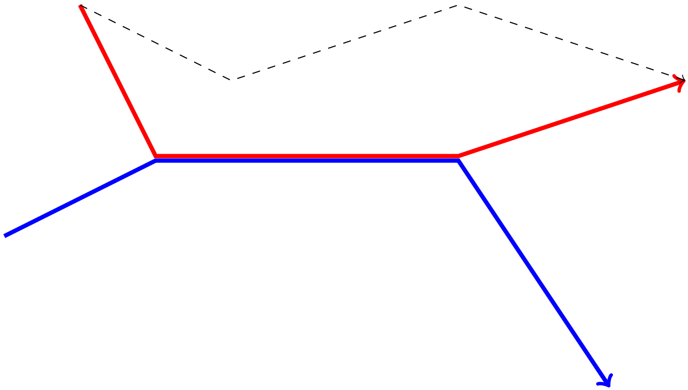


A “Theorem”

Mergings $\Rightarrow$ Congestions a.s.!

A "Proof"

Motivation in Transportation Networks



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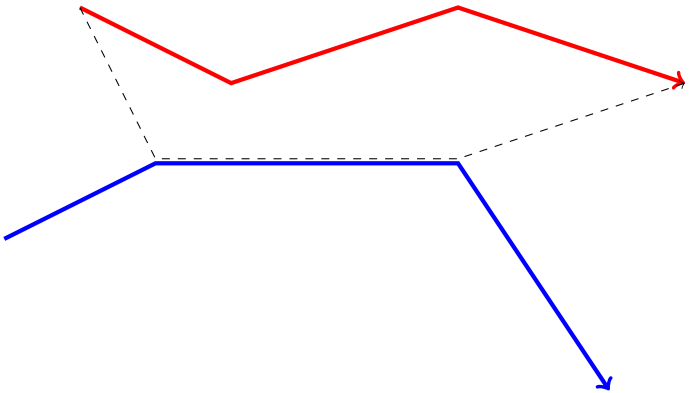

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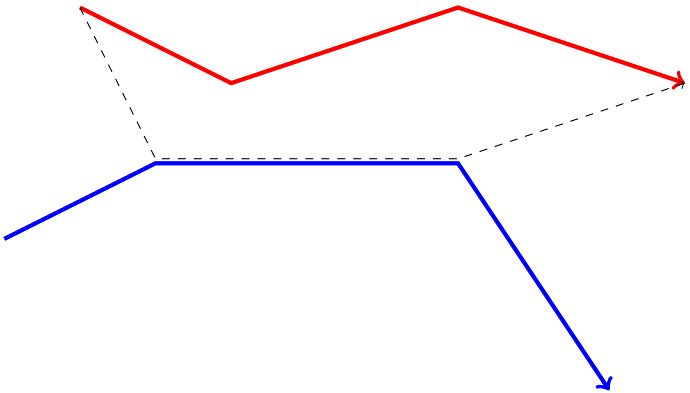
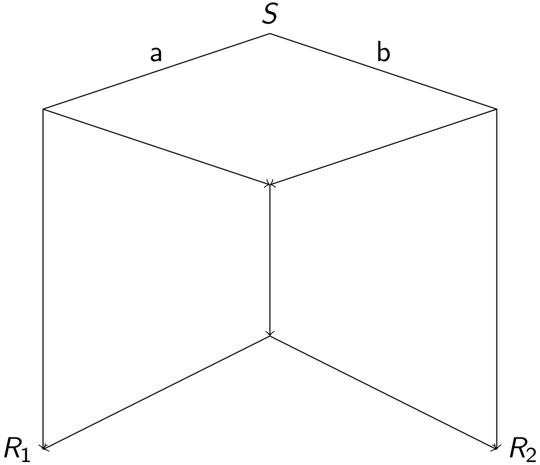


Minimizing Mergings ⇒ Maximizing Throughput!

Motivation in Computer Networks



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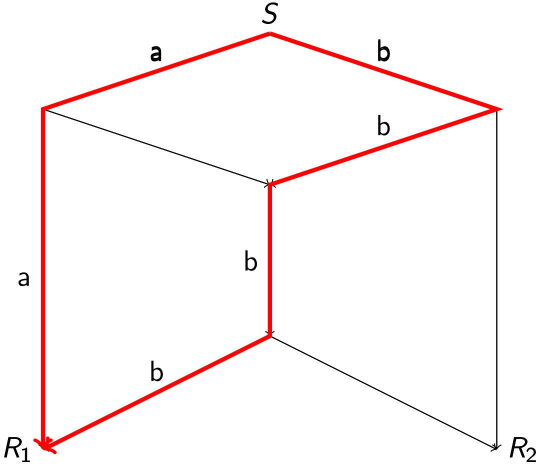

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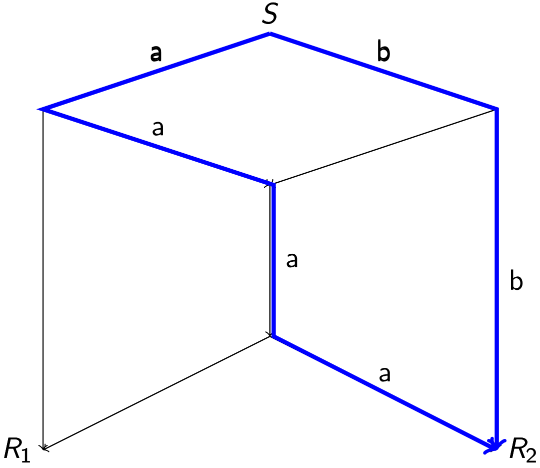

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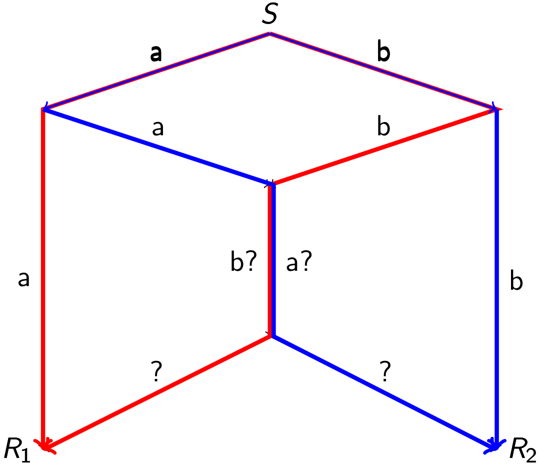

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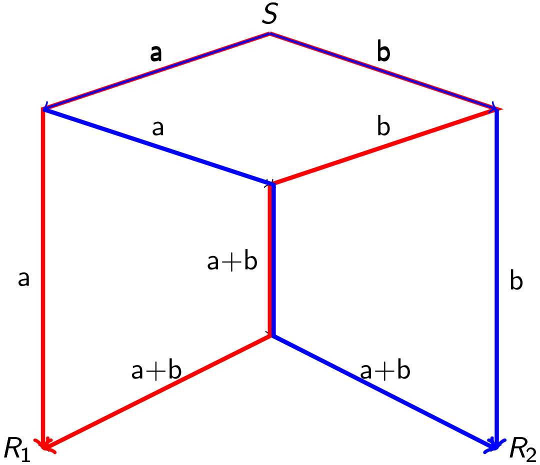

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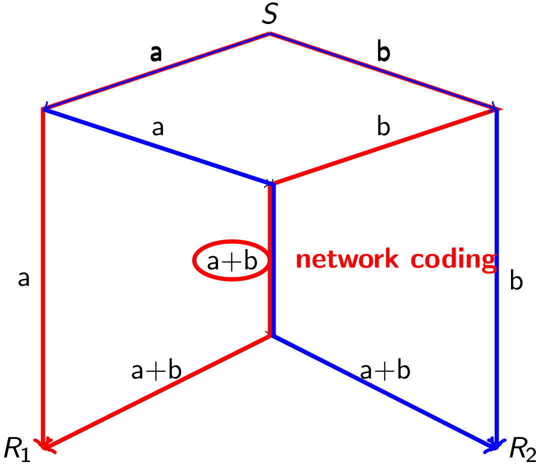

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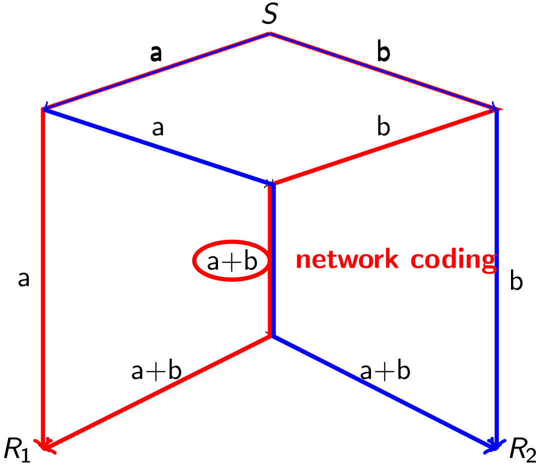
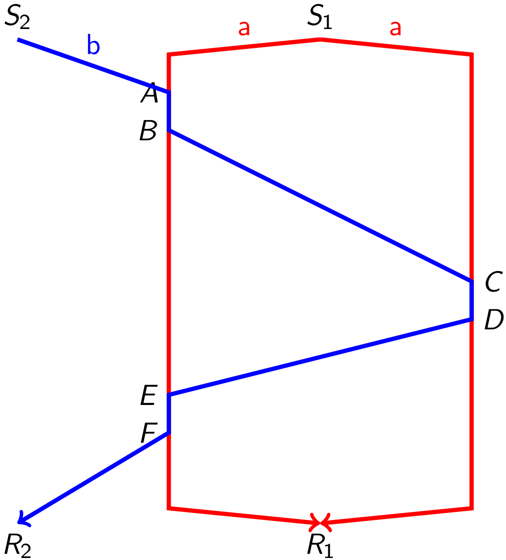


Minimizing Mergings ⇒ Minimizing Encoding Complexity!

Motivation in Computer Networks (cont'd)



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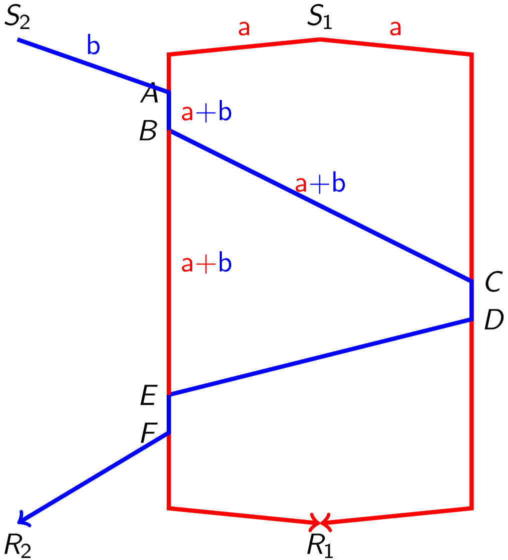

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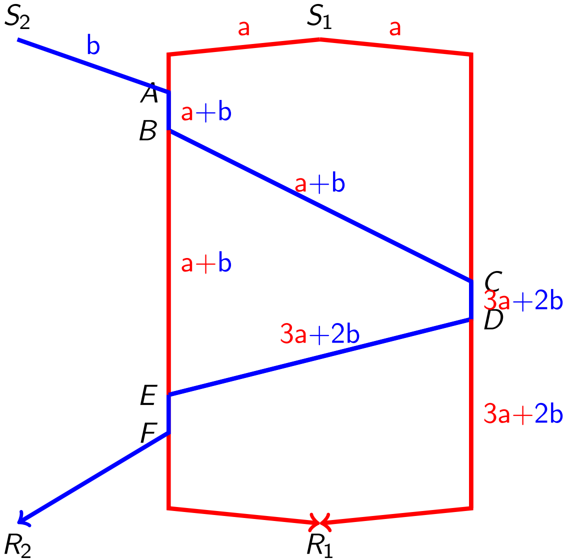

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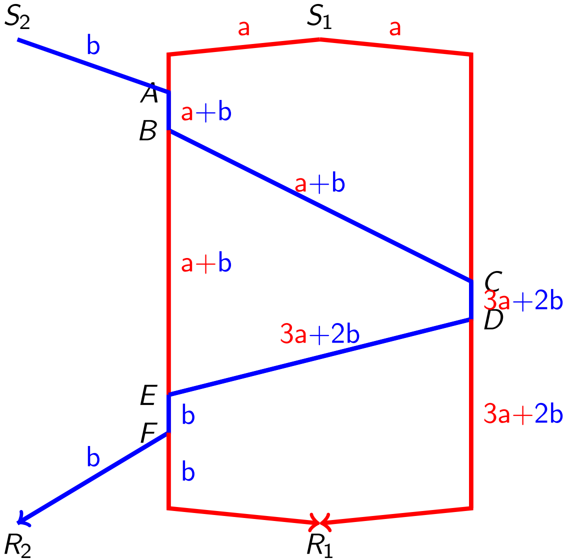

Karl Menger

Menger's Theorem

Theorem (Menger, 1927)

Consider a directed graph $G(V, E)$. For any two vertices A, B, the maximum number of pairwise edge-disjoint directed paths from A to B in G equals the min-cut between A and B, namely the minimum number of edges in E whose deletion destroys all directed paths from A to B.

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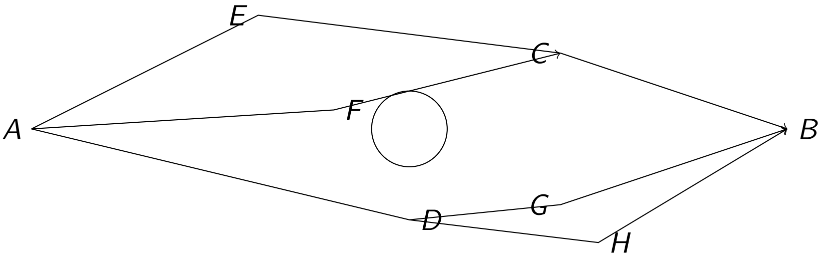
Menger's paths

A Quick Example



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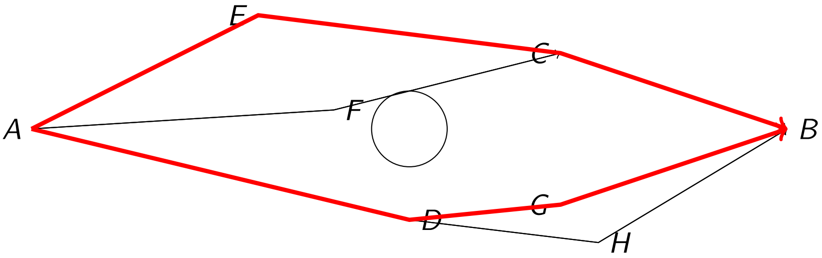

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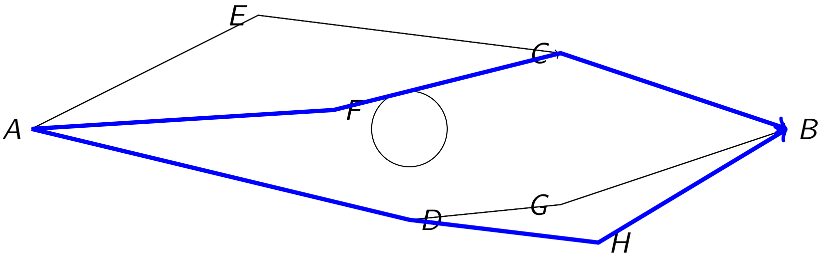

We are interested in ...

Mergings

among different groups of Menger's paths
in networks with multiple sources and sinks.

Rerouting

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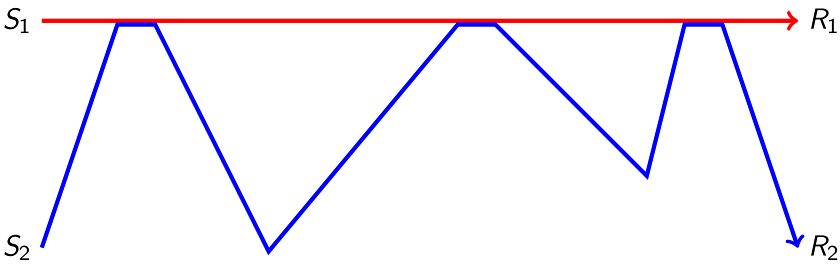

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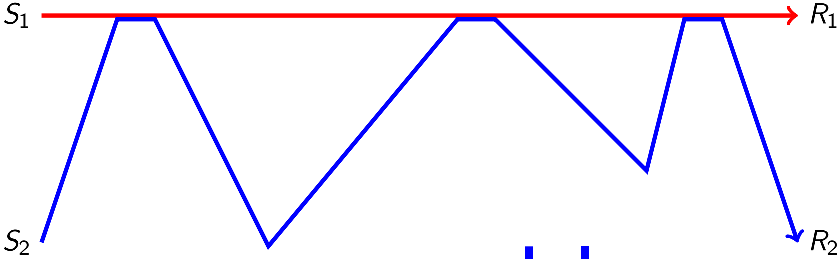

reroutable

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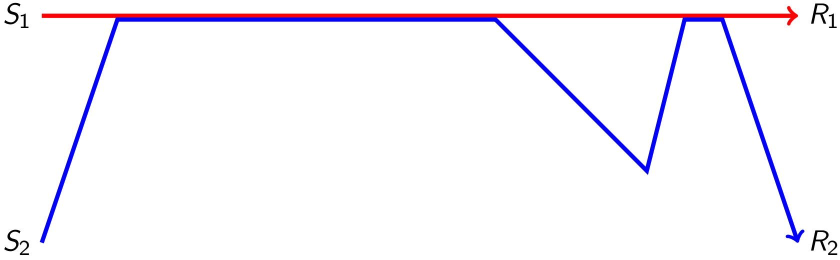
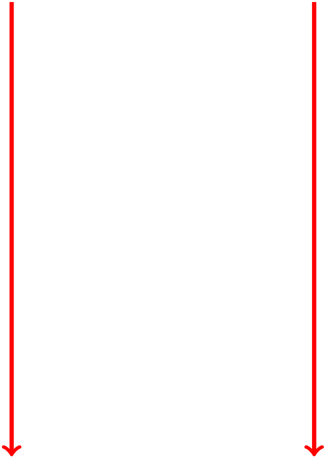


Rerouting

Rerouting (cont'd)



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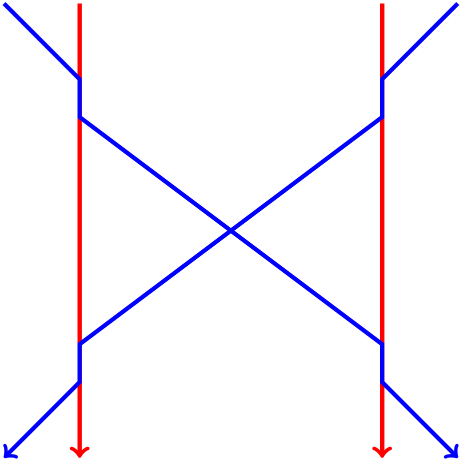

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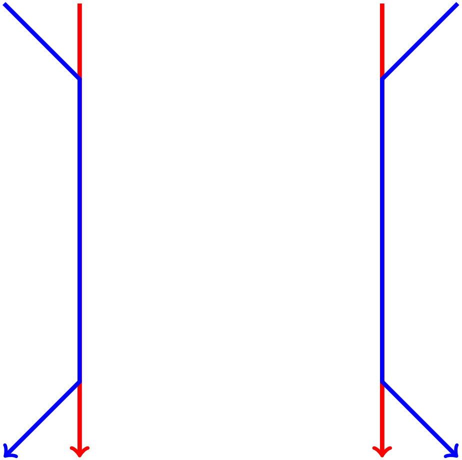

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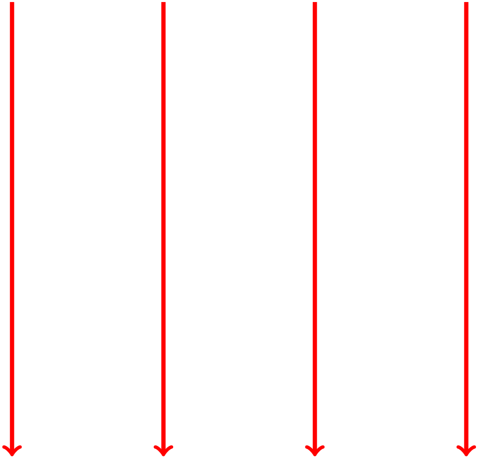

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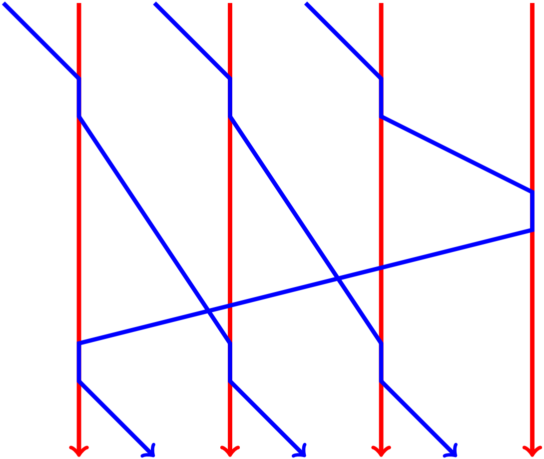

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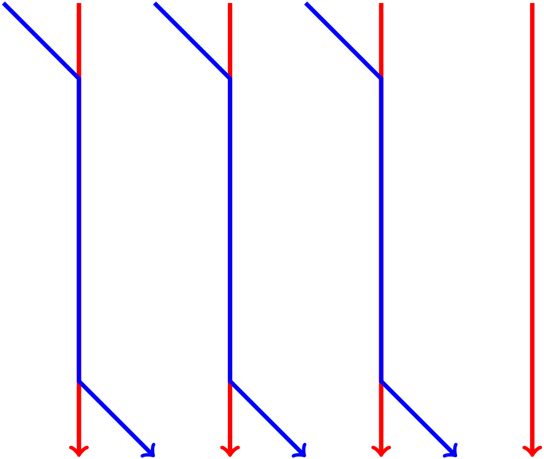

Network Model and Notations

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- $S_1, S_2, \cdots, S_n$: sources.
- $R_1, R_2, \cdots, R_n$: distinct sinks.
- c_i: the min-cut between S_i and R_i.
- $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,c_i}\}$: a set of Menger's paths from S_i to R_i.

$\mathcal{M}^*(c_1, c_2, \cdots, c_n)$

Assume that all sources are identical.

- $M^*(G)$:the minimum number of mergings over all possible Menger's path sets α_i's, $i = 1, 2, \cdots, n$.
- $\mathcal{M}^*(c_1, c_2, \cdots, c_n)$: the supremum of $M^*(G)$ over all possible choices of such G.

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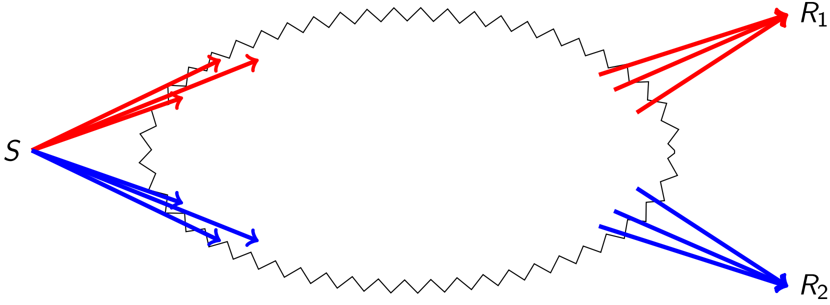

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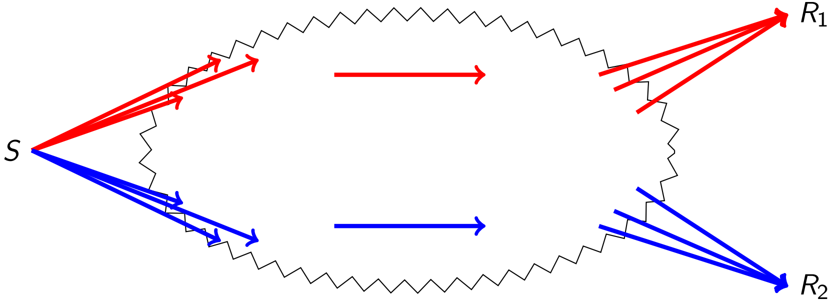

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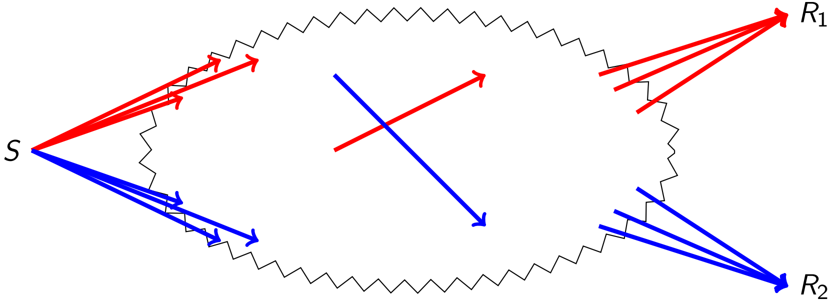

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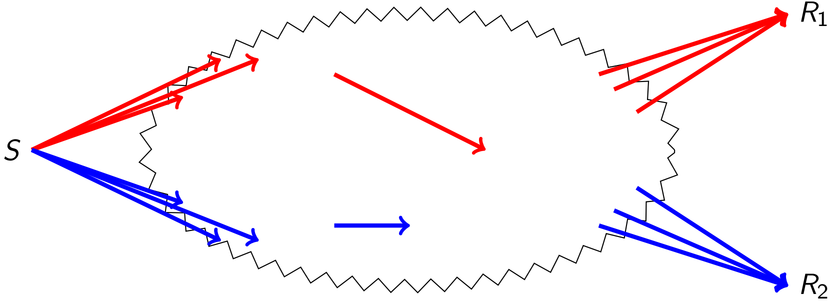

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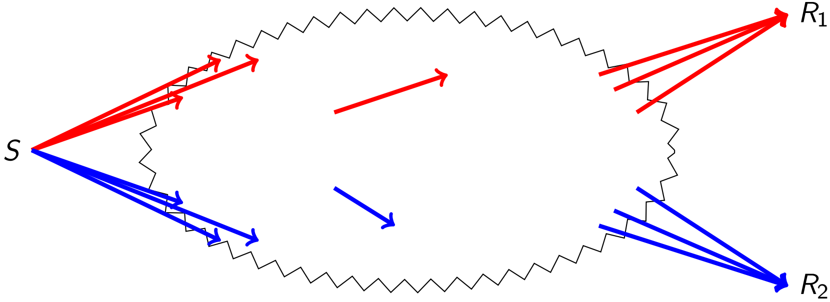

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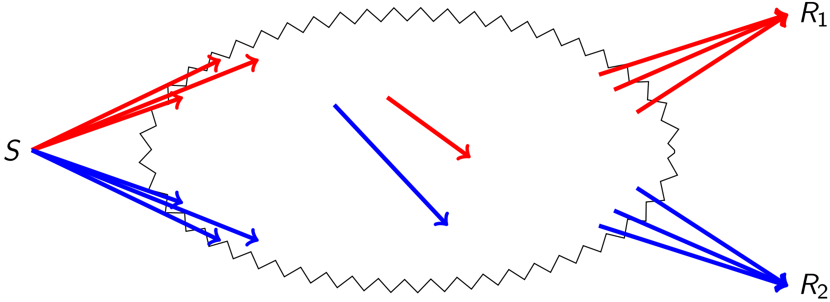

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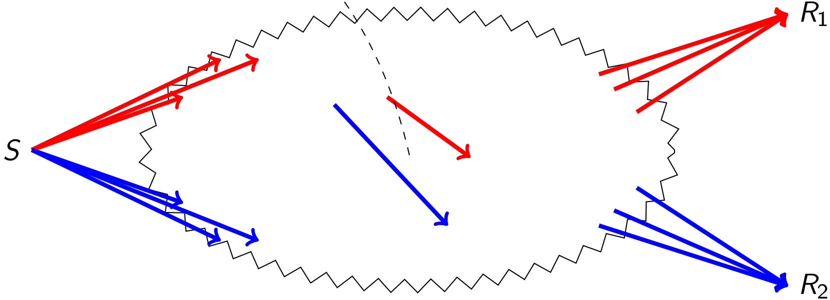

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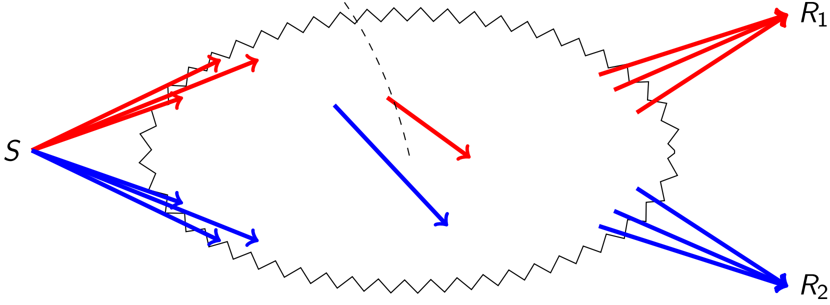
best decision

- $M^*(G)$: the minimum number of mergings over all possible Menger's path sets α_i's, $i = 1, 2, \cdots, n$.

worst performance

- $\mathcal{M}^*(c_1, c_2, \cdots, c_n)$: the supremum of $M^*(G)$ over all possible choices of such G.



$\mathcal{M}(c_1, c_2, \cdots, c_n)$

Assume that all sources are distinct.

- $M(G)$: the minimum number of mergings over all possible Menger's path sets α_i's, $i = 1, 2, \cdots, n$.
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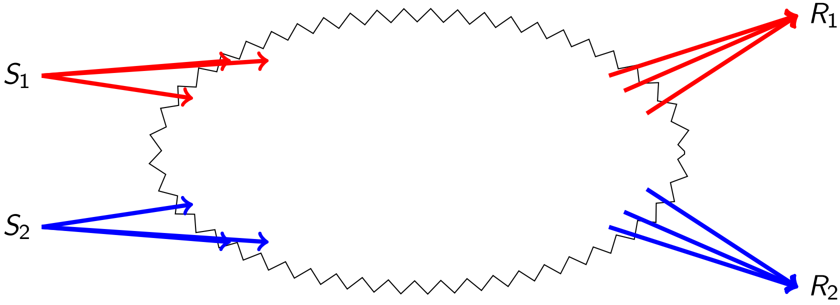

Previous Work

It was first conjectured by Tavory, Feder and Ron that $\mathcal{M}(c_1, c_2, \cdots, c_n)$ is finite. More specifically the authors proved that if $\mathcal{M}(c_1, c_2)$ is finite for all c_1, c_2, then $\mathcal{M}(c_1, c_2, \cdots, c_n)$ is finite as well.

As for $\mathcal{M}^*$, Fragouli and Soljanin use the idea of "subtree decomposition" to first prove that

$$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) = n - 1.$$

It was first shown Langberg, Sprintson and Bruck that $\mathcal{M}^*(c_1, c_2)$ is finite for all c_1, c_2, and subsequently $\mathcal{M}^*(c_1, c_2, \cdots, c_n)$ is finite all $c_1, c_2, \cdots, c_n$.

Main Results

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- For fixed $c_1, c_2, \cdots, c_n$, $\mathcal{M}(c_1, c_2, \cdots, c_n)$ and $\mathcal{M}^*(c_1, c_2, \cdots, c_n)$ are always finite.
- And as functions of $c_1, c_2, \cdots, c_n$, they have interesting properties.
- We give exact values of and tighter bounds on $\mathcal{M}$ and $\mathcal{M}^*$ with certain parameters.

Some Remarks

When $n = 1$, Ford-Fulkerson algorithm can find the min-cut and a set of Menger's path between S_1 and R_1 in polynomial time.

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The LDP (Link Disjoint Problem) asks if there are two edge-disjoint paths from S_1, S_2 to R_1, R_2, respectively. The fact that the LDP problem is NP-complete suggests the intricacy of the problem when $n \geq 2$.

Outline of the Proof

Lemma

For any c_1, c_2,

$$\mathcal{M}(c_1, c_2) \leq c_1 c_2 (c_1 + c_2)/2.$$

Theorem

For any $c_1, c_2, \cdots, c_n$, *we have*

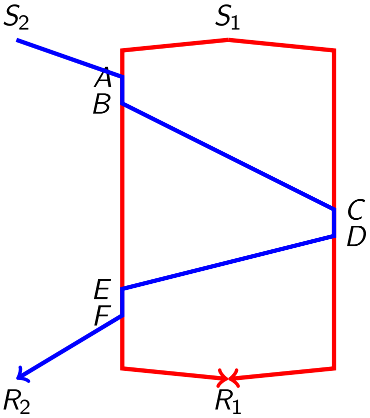
$$\mathcal{M}(c_1, c_2, \cdots, c_n) \leq \sum_{i<j} \mathcal{M}(c_i, c_j).$$

Observation

Note that for any $c_1, c_2, \cdots, c_n$, we always have

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) \leq \mathcal{M}(c_1, c_2, \cdots, c_n)$$

Proof of the Lemma [1]



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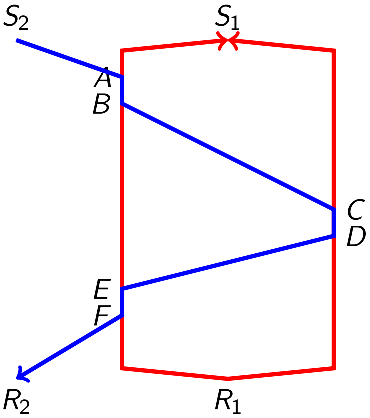

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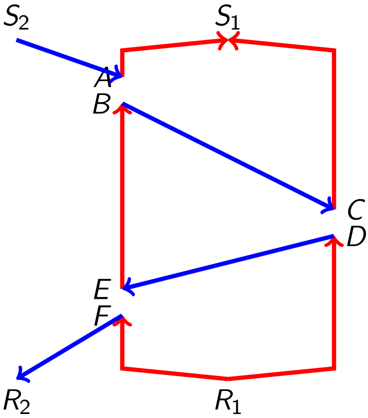

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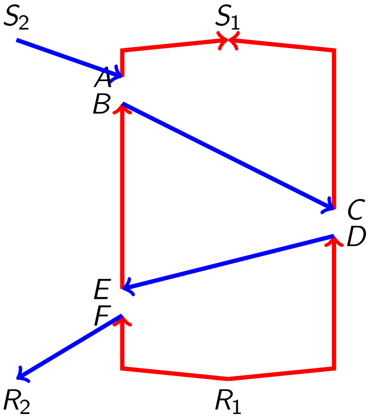

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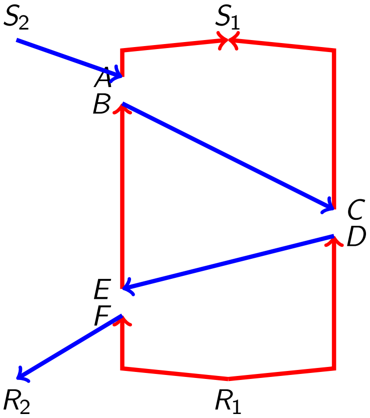

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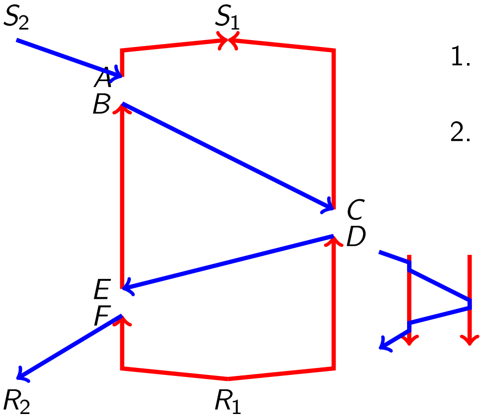

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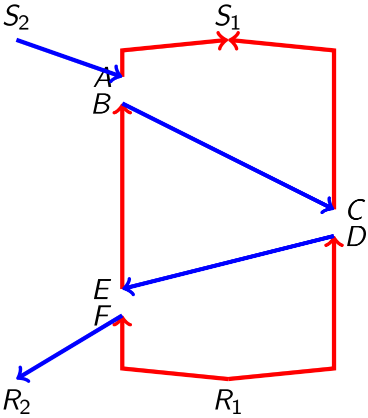

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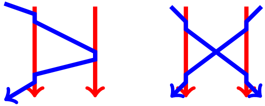

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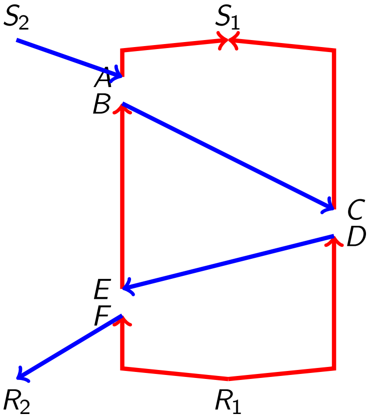

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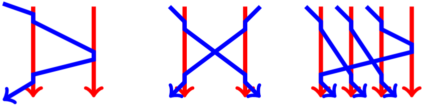

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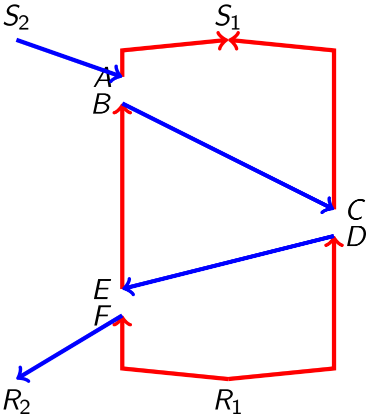
Proof of the Lemma [1]



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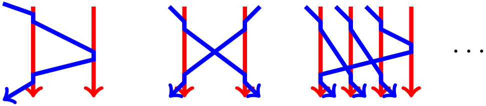

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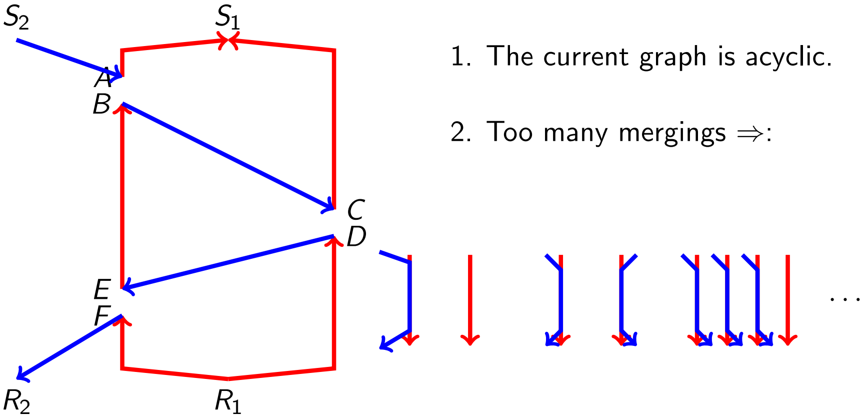

1. The current graph is acyclic.
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[1] the proof is inspired by the work of Langberg, Sprintson and Bruck

Properties of $\mathcal{M}^*$

Proposition

For $c_1 \leq c_2 \leq \cdots \leq c_n$, if $c_1 + c_2 + \cdots + c_{n-1} \leq c_n$, then

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) = \mathcal{M}^*(c_1, c_2, \cdots, c_{n-1}, c_1 + c_2 + \cdots + c_{n-1}).$$

Proposition

For $c_1 = 1 \leq c_2 \leq \cdots \leq c_n$, we have

$$\mathcal{M}^*(c_1, c_2, \cdots, c_n) = \mathcal{M}^*(c_2, \cdots, c_{n-1}, c_n).$$

Proposition

For $n_1 \leq n_2 \leq \cdots \leq n_k$,

$$\mathcal{M}^*(n_1, n_2, \ldots, n_k) \geq \sum_{i=1}^{k-1} \mathcal{M}^*(n_i, n_i).$$

Properties of $\mathcal{M}$

Proposition

For any $c_{1,0}, c_{1,1}, c_2$, we have

$$\mathcal{M}(c_{1,0} + c_{1,1}, c_2) \geq \mathcal{M}(c_{1,0}, c_2) + \mathcal{M}(c_{1,1}, c_2).$$

Proposition

For any $c_1, c_2, \cdots, c_n$ and any fixed k with $1 \leq k \leq n$, we have

$$\mathcal{M}(c_1, c_2, \cdots, c_n) \geq \sum_{i \leq k, j \geq k+1} \mathcal{M}(c_i, c_j).$$

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Properties of $\mathcal{M}$ (cont'd)

Proposition

For any $m \leq n$, we have

$$\mathcal{M}(m,n) \leq U(m,n) + V(m,n) + m - 2,$$

where

$$U(m,n) = \sum_{j=1}^{m-1} (\mathcal{M}(j, m-1) + 1 + \mathcal{M}(m-j, n)) + \mathcal{M}(m, m-1) + 1,$$

and

$$V(m,n) = \mathcal{M}(m, n-1) + \sum_{j=1}^{m-1} (\mathcal{M}(j,n) + 1 + \mathcal{M}(m-j, n)) - \mathcal{M}(1, n).$$

Properties of $\mathcal{M}$ (cont'd)

Proposition

For any fixed k, there exists a positive constant C_k such that for all n,

$$\mathcal{M}(k,n) \le C_k n.$$

Proof.

Properties of $\mathcal{M}$ (cont'd)

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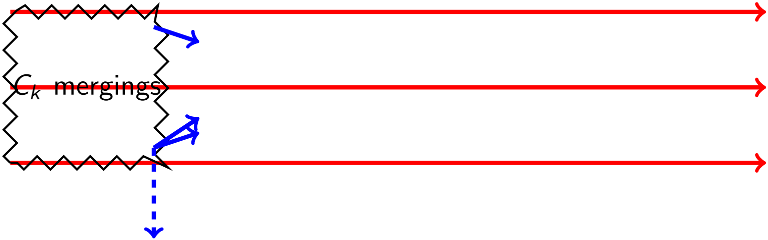

□

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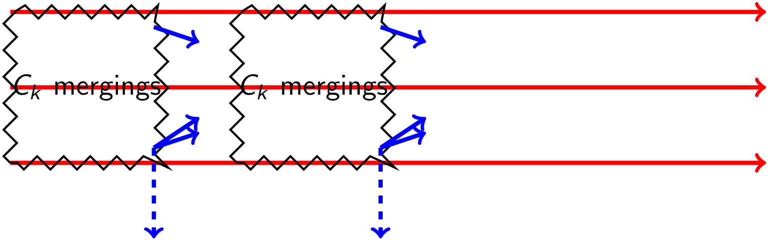

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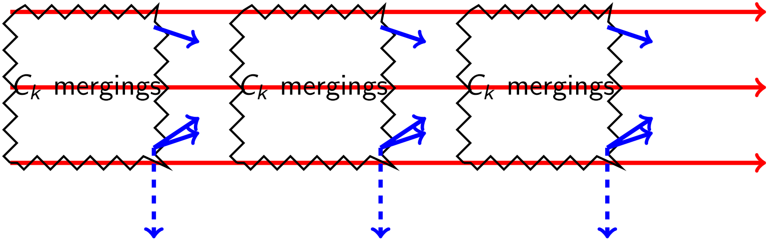


□

Tighter Bounds

It has been established [Langberg et al.] that

$$n(n-1)/2 \leq \mathcal{M}^*(n,n) \leq n^3.$$

Next, we give tighter bounds on $\mathcal{M}^*(n,n)$ and $\mathcal{M}(m,n)$.

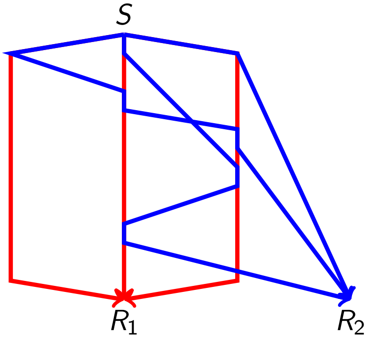
Bounds on $\mathcal{M}^*$

Proposition

$$(n-1)^2 \leq \mathcal{M}^*(n,n) \leq \left\lceil \frac{n}{2} \right\rceil (n^2 - 4n + 5).$$

Proof.

A graph showing $\mathcal{M}^*(3,3) \geq 4$.



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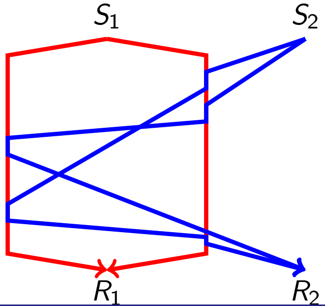
Bounds on $\mathcal{M}$

Proposition

$$2mn - m - n + 1 \leq \mathcal{M}(m, n) \leq (m+n-1) + (mn-2)\left\lfloor \frac{m+n-2}{2} \right\rfloor.$$

Proof.

A graph showing $\mathcal{M}(2,2) \geq 5$.



Exact Values

- $$\mathcal{M}^*(1,1) = 0.$$
- $$\mathcal{M}^*(2,2) = 1.$$
- $$\mathcal{M}^*(3,3) = 4.$$
- $$\mathcal{M}^*(4,4) = 9.$$
- $$\mathcal{M}^*(5,5) = 16.$$

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$$\mathcal{M}^*(m,m) = (m-1)^2?$$

$$\mathcal{M}^*(6,6) = 27!$$

Exact Values

- $$\mathcal{M}(1, n) = n.$$
- $$\mathcal{M}(2, n) = 3n - 1.$$
- $$\mathcal{M}(3, 3) = 13.$$
- $$\mathcal{M}(3, 4) = 18.$$
- $$\mathcal{M}(3, 5) = 23.$$
- $$\mathcal{M}(3, 6) = 28.$$
- $$\mathcal{M}(4, 4) = 27.$$

Exact Values

- $$\mathcal{M}^*(\underbrace{1, 1, \cdots, 1}_{n}) = 0.$$
- $$\mathcal{M}^*(\underbrace{2, 2, \cdots, 2}_{n}) = n - 1.$$
- $$\mathcal{M}^*(2, m, m) = 1 + (m-1)^2, m = 2, 3, 4, 5.$$
- $$\mathcal{M}^*(m, m, m) = 2(m-1)^2, m = 1, 2, 3, 4.$$
- $$\mathcal{M}^*(3, 4, 4) = 13.$$
- For $m \leq n \leq p$ and $(m, n) \leq (3, 4)$ or $(2, 5)$
$$\mathcal{M}^*(m, n, p) = \mathcal{M}^*(m, n, n).$$

Exact Values

- $$\mathcal{M}(\underbrace{1, 1, \cdots, 1}_{k}) = \left\lfloor \frac{k^2}{4} \right\rfloor.$$

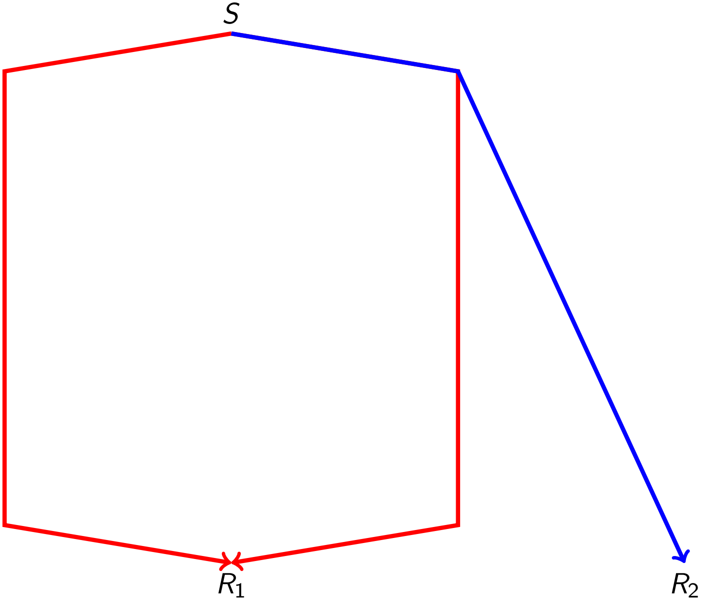
- $$\mathcal{M}(\underbrace{1, \ldots, 1}_{k}, 2) = \begin{cases} 3k - 1 & \text{if } k \leq 6, \\ \lfloor \frac{k^2}{4} \rfloor + k + 2 & \text{if } k > 6. \end{cases}$$

- $$\mathcal{M}(1, 1, n) = 2n + 1.$$

- $$\mathcal{M}(1, 2, n) = \begin{cases} 4n & \text{if } n = 2, 3, \\ 4n + 1 & \text{if } n = 1 \text{ or } n \geq 4. \end{cases}$$

- $$\mathcal{M}(2, 2, 2) = 11, \mathcal{M}(1, 3, 3) = 17, \mathcal{M}(2, 2, 3) = 18.$$

The “Finiteness” Result Does Not Hold for Cyclic Graphs



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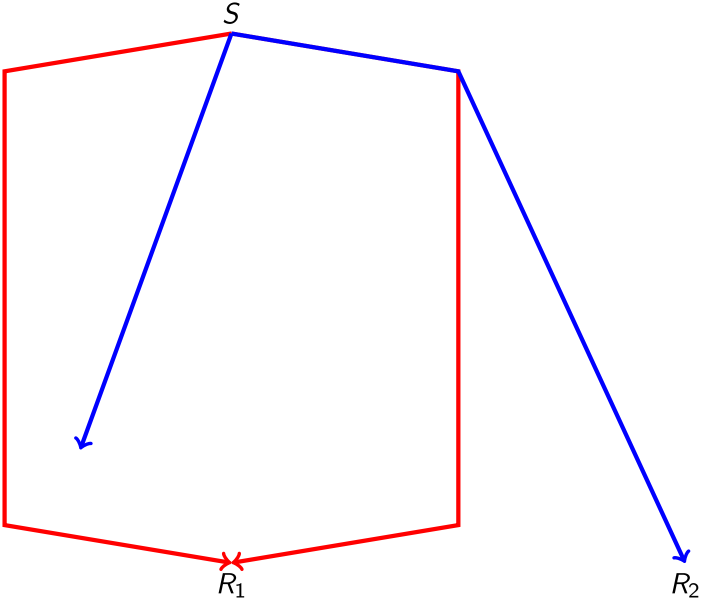

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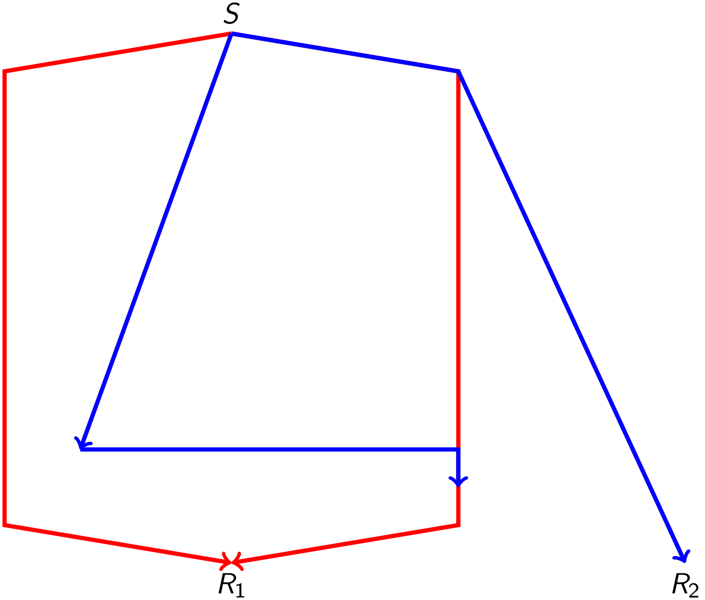

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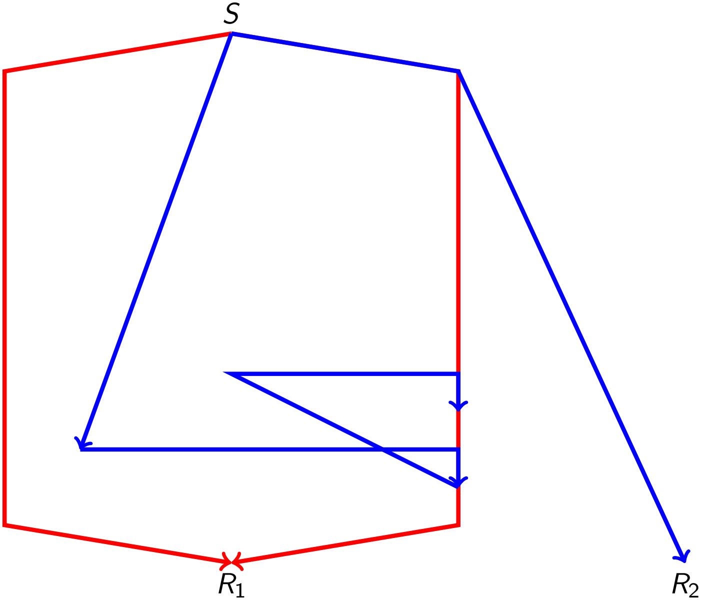

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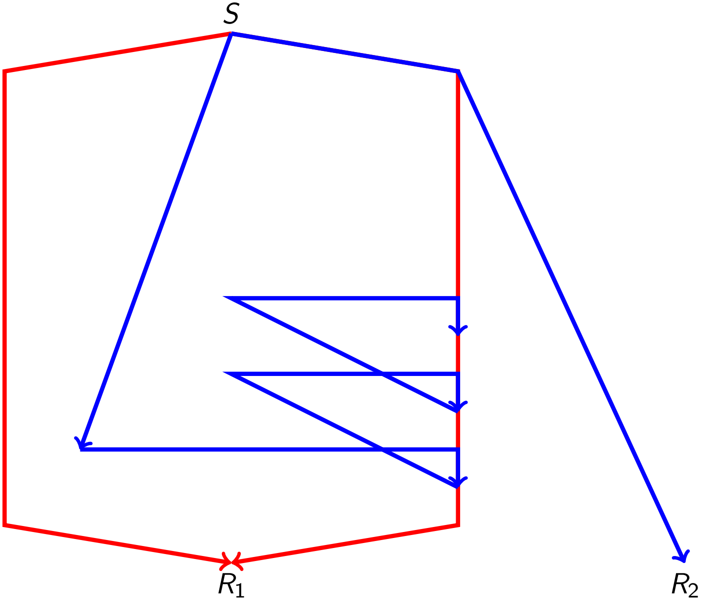

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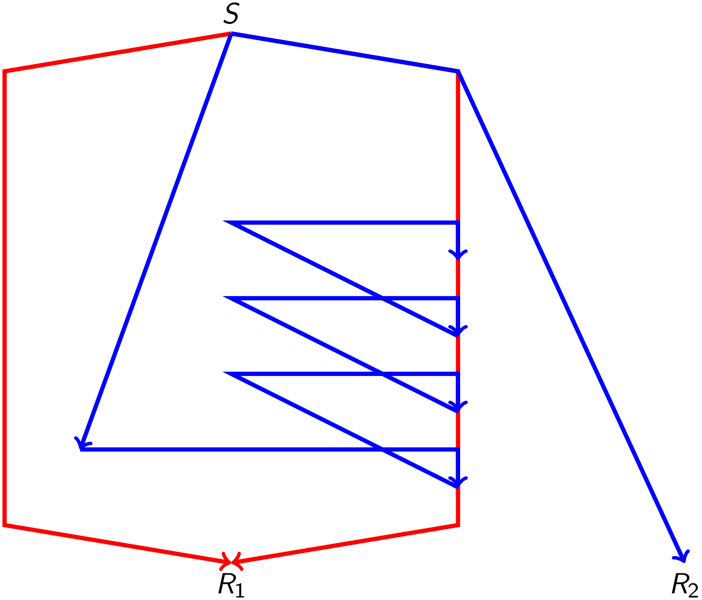

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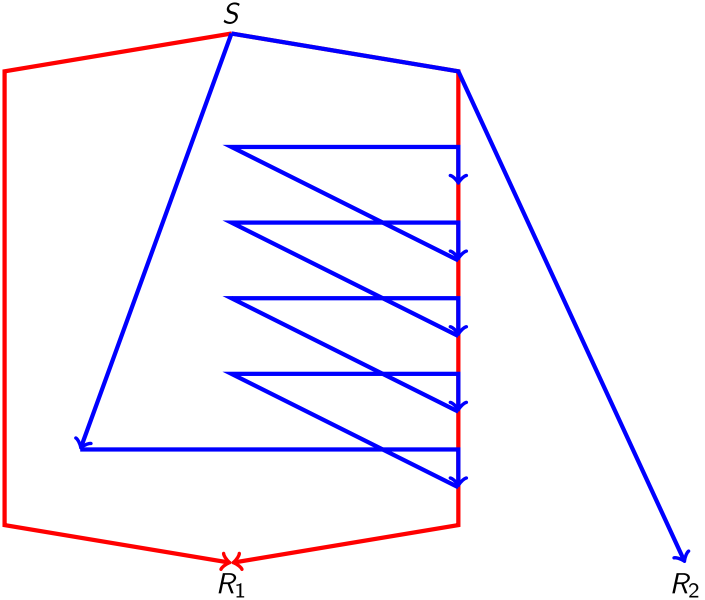

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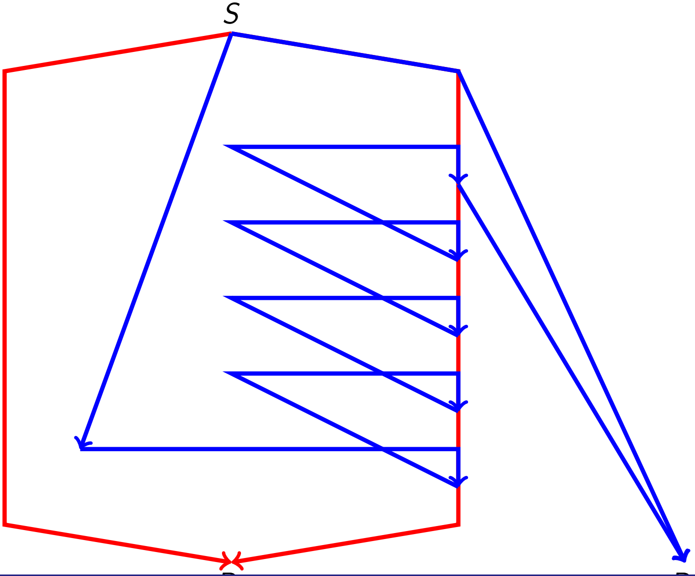

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Thank You!