New Results on Interference Channels with Three User Pairs

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Interference channel
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Interference channel

- Cellular networks
- Ad-hoc wireless networks
- Wireline: DSL
Interference channel

- Capacity region not known in general, even for two user pairs
- Best known scheme for two user pairs: (Han–Kobayashi 81)
- How to extend the Han–Kobayashi scheme to more user pairs?
Interference channel

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Interference channel

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Best known scheme for two user pairs: (Han–Kobayashi 81)

How to extend the Han–Kobayashi scheme to more user pairs?
Deterministic interference channels

- General interference channels contain two adverse effects
  - Channel noise
  - Interference
Deterministic interference channels

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▶ Deterministic interference channels
  - No noise in the channel
  - Focus on signal interaction
Deterministic interference channels

- General interference channels contain two adverse effects
  - Channel noise
  - Interference

- **Deterministic interference channels**
  - No noise in the channel
  - Focus on signal interaction
  - Motivated by
    - Growing user density in current wireless networks
    - High-SNR regime of Gaussian channels *(Bresler–Tse 08)*
    - Study adverse effects separately
Deterministic interference channel (3-DIC)

\[ M_1 \to X_1 \quad Y_1 \to \hat{M}_1 \]
\[ M_2 \to X_2 \quad Y_2 \to \hat{M}_2 \]
\[ M_3 \to X_3 \quad Y_3 \to \hat{M}_3 \]
Deterministic interference channel (3-DIC)

$M_1 \rightarrow X_1 \rightarrow Y_1 \rightarrow \hat{M}_1$

$M_2 \rightarrow X_2 \rightarrow Y_2 \rightarrow \hat{M}_2$

$M_3 \rightarrow X_3 \rightarrow Y_3 \rightarrow \hat{M}_3$

$X_{11} \rightarrow f_1 \rightarrow Y_1$

$X_{21} \rightarrow h_1 \rightarrow S_1 \rightarrow Y_1$

$X_{31} \rightarrow h_1 \rightarrow S_1 \rightarrow Y_1$
Deterministic interference channel (3-DIC)

Injectivity: $h_k$ and $f_k$ are one-to-one in each argument

For $f_1$: $H(X_{11}) = H(Y_1 | S_1)$ and $H(S_1) = H(Y_1 | X_{11})$
Define \((2^{nR_1}, 2^{nR_2}, 2^{nR_3}, n)\) code, probability of error, achievability of \((R_1, R_2, R_3)\), and the capacity region in the usual way.
Deterministic interference channel (3-DIC)
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- Why this deterministic model?
  - 2-pair deterministic interference channel \textit{(El Gamal–Costa 82)}
  - Fully invertible 3-DIC \textit{(Gou–Jafar 09)}
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Deterministic interference channel (3-DIC)

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- Capacity region is not known in general

- We find a new achievable rate region
  - Includes previously known bounds
  - Naturally extends Han–Kobayashi
Two user pairs \textit{(El Gamal–Costa 82)}

\[
\begin{align*}
M_1 & \rightarrow X_1 \quad f_1 \quad Y_1 \rightarrow \hat{M}_1 \\
M_2 & \rightarrow X_2 \quad f_2 \quad Y_2 \rightarrow \hat{M}_2
\end{align*}
\]
Two user pairs (El Gamal–Costa 82)

Coding strategies:

- Treat interference as noise
Two user pairs \((El \text{ Gamal–Costa 82})\)

\[
\begin{align*}
M_1 &\rightarrow X_1 \quad \xrightarrow{f_1} \quad Y_1 \rightarrow \hat{M}_1 \\
M_2 &\rightarrow X_2 \quad \xrightarrow{f_2} \quad Y_2 \rightarrow \hat{M}_2
\end{align*}
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Coding strategies:
- Treat interference as noise
- Decode both messages
Two user pairs \((El \ Gamal–Costa \ 82)\)

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\begin{align*}
M_1 &\rightarrow X_1 & \quad f_1 &\rightarrow Y_1 &\rightarrow \hat{M}_1 \\
M_2 &\rightarrow X_2 & \quad f_2 &\rightarrow Y_2 &\rightarrow \hat{M}_2 
\end{align*}
\]

**Coding strategies:**

- Treat interference as noise
- Decode both messages
- Hybrid: partly decode, partly treat as noise \((Han–Kobayashi \ 81)\)
  - Rate splitting and superposition coding
  - Achieves entire capacity region
Much less is known

K ≥ 3 user pairs
$K \geq 3$ user pairs

- Much less is known

- Receivers are impaired by the **joint effect** of interferers
  - Partially decoding interfering messages is not appropriate
Much less is known

Receivers are impaired by the joint effect of interferers
  - Partially decoding interfering messages is not appropriate

Interference alignment
  (Maddah-Ali et al. 08, Cadambe–Jafar 08)
  - Constrain combined interference to a subspace
  - Disregard that subspace
  - Treat interference as noise
$K \geq 3$ user pairs

- Much less is known

- Receivers are impaired by the joint effect of interferers
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- Interference alignment
  (Maddah-Ali et al. 08, Cadambe–Jafar 08)
  - Constrain combined interference to a subspace
  - Disregard that subspace
  - Treat interference as noise

- We are interested in more general coding schemes
Two aspects of interference channels

\[ M_1 \rightarrow \text{Enc 1} \rightarrow X_1^n \rightarrow \text{Interference Channel} \rightarrow Y_1^n \rightarrow \text{Dec 1} \rightarrow \hat{M}_1 \]

\[ M_2 \rightarrow \text{Enc 2} \rightarrow X_2^n \rightarrow \text{Interference Channel} \rightarrow Y_2^n \rightarrow \text{Dec 2} \rightarrow \hat{M}_2 \]

\[ M_3 \rightarrow \text{Enc 3} \rightarrow X_3^n \rightarrow \text{Interference Channel} \rightarrow Y_3^n \rightarrow \text{Dec 3} \rightarrow \hat{M}_3 \]
Two aspects of interference channels

Multiple Access Channel

But: receiver is interested in only one message
Two aspects of interference channels

Broadcast channel

But: transmitter has a message for only one receiver
Talk outline

- Receiver-centric (MAC) aspect
  - Interference decoding

- Transmitter-centric (BC) aspect
  - Communication with disturbance constraints

- Combine the two aspects
  - Capacity inner bound for 3-DIC

- Extension to noisy channels
Receiver-centric aspect

But: receiver is interested in only one message
Receiver-centric aspect

- Use simple point-to-point codes at the transmitters
- **Interference decoding**
  Receivers decode own message jointly with combined interference
  Simultaneous non-unique decoding
Interference-decoding inner bound \((B./El\ Gamal,\ ISIT\ 2010)\)

**Theorem**

\[
\mathcal{R}_{ID} = \bigcup_{p} \mathcal{R}_1(p) \cap \mathcal{R}_2(p) \cap \mathcal{R}_3(p),
\]

where \((Q, X_1, X_2, X_3) \sim p = p(q)p(x_1|q)p(x_2|q)p(x_3|q),\) is an inner bound to the capacity region of the 3-DIC.
Theorem

\[ R_{ID} = \bigcup_{p} R_1(p) \cap R_2(p) \cap R_3(p), \]

where \( (Q, X_1, X_2, X_3) \sim p = p(q)p(x_1|q)p(x_2|q)p(x_3|q) \), is an inner bound to the capacity region of the 3-DIC.

- \( R_1(p) \) is the set of \((R_1, R_2, R_3)\) that satisfy

  \[ R_1 < H(X_{11} | Q), \]

  \[ R_1 + \min\{R_2, H(X_{21} | Q)\} < H(Y_1 | X_{31}, Q), \]

  \[ R_1 + \min\{R_3, H(X_{31} | Q)\} < H(Y_1 | X_{21}, Q), \]

  \[ R_1 + \min\{R_2 + R_3, H(S_1 | Q), R_2 + H(X_{31} | Q), H(X_{21} | Q) + R_3\} < H(Y_1 | Q) \]

- Likewise, \( R_2 \) and \( R_3 \)
**Interference-decoding inner bound**  
*(B./El Gamal, ISIT 2010)*

**Theorem**

\[ \mathcal{R}_{ID} = \bigcup_{p} \mathcal{R}_1(p) \cap \mathcal{R}_2(p) \cap \mathcal{R}_3(p), \]

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**Remarks:**

- Generally larger than interference-as-noise inner bound
  - Interference alignment
Interference-decoding inner bound \((B./\text{El Gamal, ISIT 2010})\)

**Theorem**

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is an inner bound to the capacity region of the 3-DIC.

**Remarks:**

- Generally larger than interference-as-noise inner bound
  - Interference alignment
- Achieves capacity in some cases
  - Sum capacity of 3-user \(Q\)-ary extension channel \((B. \text{ et al. ISIT 2009})\)
  - Strong interference, invertible \(h_k\)
General shape of $\mathcal{R}_1$

- $\mathcal{R}_1$ is unbounded in $R_2$ and $R_3$
  - $R_1 > 0$ regardless of $R_2$, $R_3$
- As $R_2$, $R_3$ decrease, $S_1$ becomes more structured
  - Helps increase $R_1$

\[
R_1 < H(X_{11} | Q),
\]
\[
R_1 + \min\{R_2, H(X_{21} | Q)\} < H(Y_1 | X_{31}, Q),
\]
\[
R_1 + \min\{R_3, H(X_{31} | Q)\} < H(Y_1 | X_{21}, Q),
\]
\[
R_1 + \min\{R_2 + R_3, H(S_1 | Q), R_2 + H(X_{31} | Q), H(X_{21} | Q) + R_3\} < H(Y_1 | Q)
\]
**Saturation effect**

\[ R_1 < H(X_{11} \mid Q), \]
\[ R_1 + \min\{R_2, H(X_{21} \mid Q)\} < H(Y_1 \mid X_{31}, Q), \]
\[ R_1 + \min\{R_3, H(X_{31} \mid Q)\} < H(Y_1 \mid X_{21}, Q), \]
\[ R_1 + \min\{R_2 + R_3, H(S_1 \mid Q), R_2 + H(X_{31} \mid Q), \]
\[ H(X_{21} \mid Q) + R_3\} < H(Y_1 \mid Q) \]
Proof of achievability

- **Codebook generation**
  
  Fix $p(x_1)p(x_2)p(x_3)$
  
  Generate $x_1^n(m_1) \sim \prod_{i=1}^n p_{X_1}(x_{1i})$, for $m_1 \in [1 : 2^{nR_1}]$
  
  Repeat likewise for other users
Proof of achievability

- **Codebook generation**
  
  Fix $p(x_1)p(x_2)p(x_3)$
  
  Generate $x_1^n(m_1) \sim \prod_{i=1}^n p(x_1(x_i))$, for $m_1 \in [1:2^{nR_1}]$
  
  Repeat likewise for other users
  
  (This induces $x_{12}^n(m_1)$, $s_1^n(m_2, m_3)$, $y_2^n(m_1, m_2, m_3)$, etc.)
Proof of achievability

- **Codebook generation**
  Fix \( p(x_1)p(x_2)p(x_3) \)
  Generate \( x_1^n(m_1) \sim \prod_{i=1}^n p(x_1(x_i)), \) for \( m_1 \in [1 : 2^{nR_1}] \)
  Repeat likewise for other users
  (This induces \( x_{12}^n(m_1), s_1^n(m_2, m_3), y_2^n(m_1, m_2, m_3), \) etc.)

- **Encoding**
  To send \( m_1 \), transmit \( x_1^n(m_1) \). Likewise for other users
Proof of achievability

- **Codebook generation**
  Fix $p(x_1)p(x_2)p(x_3)$
  Generate $x_1^n(m_1) \sim \prod_{i=1}^n p(x_1(x_i))$, for $m_1 \in [1:2^{nR_1}]$
  Repeat likewise for other users
  (This induces $x_{12}^n(m_1)$, $s_1^n(m_2, m_3)$, $y_2^n(m_1, m_2, m_3)$, etc.)

- **Encoding**
  To send $m_1$, transmit $x_1^n(m_1)$. Likewise for other users

- **Decoding:** Simultaneous non-unique decoding
  Observe $y_1^n$. Find unique $\hat{m}_1$ such that
  \[
  (x_1^n(\hat{m}_1), s_1^n(m_2, m_3), x_{21}^n(m_2), x_{31}^n(m_3), y_1^n) \in \mathcal{T}_\varepsilon^{(n)}
  \]
  for some $m_2, m_3$
  Likewise for other users
Optimality

- Interference decoding inner bound not optimal in general
Optimality

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  ▶ But:
  
  For fixed codebook structure, the decoder structure is optimal
Optimality

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Optimality

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  ➤ But:
  For fixed codebook structure, the *decoder structure* is optimal
Optimality

- Interference decoding inner bound not optimal in general
  - But:
    For fixed codebook structure, the decoder structure is optimal
Optimality

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  - But:
    For fixed codebook structure, the decoder structure is optimal

- Receiver-centric aspect is solved
  - Simultaneous non-unique decoding makes optimal use of codebook knowledge
Optimality

- Interference decoding inner bound not optimal in general

  - But:
    For fixed codebook structure, the decoder structure is optimal

- Receiver-centric aspect is solved
  - Simultaneous non-unique decoding makes optimal use of codebook knowledge

- In fact, this holds for general $K$-pair discrete memoryless ICs
  \[(B./Kim/El Gamal, manuscript in preparation)\]
Example

\[ X_1 \rightarrow \{0, 1, 2\} \rightarrow \begin{array}{c}
0 \rightarrow 0 \\
2 \rightarrow 0 \\
1 \rightarrow 1
\end{array} \rightarrow \{0, 1\} \rightarrow \{0, \ldots, 4\} \rightarrow Y_1 \]

\[ X_2 \rightarrow \{0, 1, 2\} \rightarrow \begin{array}{c}
0 \rightarrow 0 \\
2 \rightarrow 0 \\
1 \rightarrow 1
\end{array} \rightarrow \{0, 1\} \rightarrow \{0, \ldots, 4\} \rightarrow Y_2 \]

\[ X_3 \rightarrow \{0, 1, 2\} \rightarrow \begin{array}{c}
0 \rightarrow 0 \\
2 \rightarrow 0 \\
1 \rightarrow 1
\end{array} \rightarrow \{0, 1\} \rightarrow \{0, \ldots, 4\} \rightarrow Y_3 \]
Example: Achievable rate regions

Interference as noise

Interference decoding
Summary – Receiver-centric aspect

Interference decoding

- Improves upon treating interference as noise
- Given the codebook structure, simultaneous non-unique decoding is optimal
- Fully exploits structure of combined interference signal
Transmitter-centric aspect

Broadcast channel

**But:** transmitter has a message for only one receiver
We define **disturbance-constrained communication**
Disturbance-constrained communication  \( (B./El \; Gamal, \; ISIT \; 2011) \)

\[
M \in [2^{nR}] \xrightarrow{\text{Enc}} \quad X^n \xrightarrow{\text{Dec}} \frac{1}{n} H(Z^n) \leq R_d
\]

\[
M \xrightarrow{\text{Enc}} \quad f \rightarrow \quad Z^n \xrightarrow{\text{Dec}} \hat{M}
\]
Disturbance-constrained communication \textit{(B./El Gamal, ISIT 2011)}

Rate–disturbance trade-off

- Pair $(R, R_d)$ is achievable:
  - Sequence of $(2^{nR}, n)$ codes exists with
    \[
    \lim_{n \to \infty} P(\hat{M} \neq M) = 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} H(Z^n) \leq R_d
    \]
Disturbance-constrained communication (B./El Gamal, ISIT 2011)

Rate–disturbance trade-off

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  Sequence of \((2^{nR}, n)\) codes exists with
  \[
  \lim_{n \to \infty} P(\hat{M} \neq M) = 0 \quad \lim_{n \to \infty} \frac{1}{n} H(Z^n) \leq R_d
  \]

- Rate–disturbance region \(\mathcal{R}\)
  Closure of all achievable rate–disturbance pairs
Result: Rate–disturbance region

The rate–disturbance region $\mathcal{R}$ is the set of pairs $(R, R_d) \in \mathbb{R}^2_+$ satisfying

$$R \leq H(Y)$$
$$R - R_d \leq H(Y | Z)$$

for some distribution $p(x)$.
**Result: Rate–disturbance region**

The rate–disturbance region $\mathcal{R}$ is the set of pairs $(R, R_d) \in \mathbb{R}^2_+$ satisfying

\[
R \leq H(Y) \\
R - R_d \leq H(Y \mid Z)
\]

for some distribution $p(x)$.
Result: Rate–disturbance region

The rate–disturbance region $\mathcal{R}$ is the set of pairs $(R, R_d) \in \mathbb{R}^2_+$ satisfying

\[
R \leq H(Y) \\
R - R_d \leq H(Y \mid Z)
\]

for some distribution $p(x)$

(This theorem extends to noisy channels.)
Example

\[ M \rightarrow X \rightarrow Y \rightarrow \hat{M} \]

\[ \{0, 1, 2, 3\} \rightarrow \{0, 1, 2\} \rightarrow \{0, 1\} \rightarrow Z \]

\[ 0 \rightarrow 1 \rightarrow 3 \rightarrow 2 \]
Example

\[ M \rightarrow X \quad \{0, 1, 2, 3\} \quad \rightarrow \quad \{0, 1\} \quad \rightarrow \quad Z \]

\[ Y \rightarrow \hat{M} \]

\[ 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \quad 1.2 \quad 1.4 \quad 1.6 \quad 1.8 \]

\[ 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]

\[ R_d \quad R \]

B. Bandemer (UCSD)
Achievability proof sketch

- **Rate splitting**: \( R = R_0 + R_1 \)

- **Superposition coding**:
  
  Fix \( p(x) \). This induces \( p(z)p(x|z) \).
  Generate \( 2^{nR_0} \) cloud centers \( \sim p(z) \).
  Generate \( 2^{nR_1} \) satellite codewords \( \sim p(x|z) \).
Achievability proof sketch

- **Rate splitting:** \( R = R_0 + R_1 \)

- **Superposition coding:**
  
  Fix \( p(x) \). This induces \( p(z)p(x|z) \)
  
  Generate \( 2^{nR_0} \) cloud centers \( \sim p(z) \)
  
  Generate \( 2^{nR_1} \) satellite codewords \( \sim p(x|z) \)

- **Analysis:**
  
  Side receiver distinguishes cloud centers, but not satellite codewords
Connection to interference channel

Injective deterministic **interference channel** (*El Gamal–Costa 1982*)
Connection to interference channel

Injective deterministic interference channel (El Gamal–Costa 1982)

- Optimal scheme: Han–Kobayashi with $U_1 = Z_1$, $U_2 = Z_2$
Connection to interference channel

Injective deterministic interference channel \textit{(El Gamal–Costa 1982)}

\begin{itemize}
  \item Optimal scheme: \textbf{Han–Kobayashi} with $U_1 = Z_1$, $U_2 = Z_2$
    \begin{itemize}
      \item is rate-splitting and superposition coding
    \end{itemize}
\end{itemize}
Connection to interference channel

Injective deterministic interference channel \textit{(El Gamal–Costa 1982)}

\begin{itemize}
  \item Optimal scheme: \textbf{Han–Kobayashi} with \( U_1 = Z_1, \ U_2 = Z_2 \)
    \begin{itemize}
      \item is rate-splitting and superposition coding
      \item coincides with disturbance-minimizing scheme
    \end{itemize}
\end{itemize}
Extension to two disturbance constraints

One disturbance constraint
Optimal scheme

2-pair interference channel
Han–Kobayashi scheme
Extension to two disturbance constraints

**One disturbance constraint**

Optimal scheme $\leftrightarrow$ 2-pair interference channel

Han–Kobayashi scheme

**Two disturbance constraints**

$\rightarrow$ 3-pair interference channel
Two disturbance constraints

\[ M \in [2^n R] \]

\[ \begin{align*}
    & X^n \quad \text{Enc} \quad f_1 \quad Z^n_1 \\
    & f_2 \quad Z^n_2 \\
    & g \quad Y^n \quad \text{Dec} \quad \hat{M}
\end{align*} \]

\[ \frac{1}{n} H(Z^n_2) \leq R_{d,2} \]

\[ \frac{1}{n} H(Z^n_1) \leq R_{d,1} \]
Coding scheme

\[ X \rightarrow g \rightarrow Y \]
\[ f_1 \rightarrow Z_1 \]
\[ f_2 \rightarrow Z_2 \]
Coding scheme

- Split rate: $R = R_0 + R_1 + R_2 + R_3$
- Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$
Coding scheme

Split rate $R = R_0 + R_1 + R_2 + R_3$
Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$

$u^n$ i.i.d.
Coding scheme

Split rate $R = R_0 + R_1 + R_2 + R_3$

Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$

- $u^n$ i.i.d.
- Marton $(z_1^n, z_2^n)$ given $u^n$
Coding scheme

Split rate $R = R_0 + R_1 + R_2 + R_3$
Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$

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$u^n$ i.i.d.

Marton $(z_1^n, z_2^n)$ given $u^n$
Coding scheme

Split rate $R = R_0 + R_1 + R_2 + R_3$

Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$

- $u^n$ i.i.d.
- Marton $(z_1^n, z_2^n)$ given $u^n$
- Superimpose $x^n$
Coding scheme

Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$

Split rate $R = R_0 + R_1 + R_2 + R_3$

- $u^n$ i.i.d.
- Marton $(z_1^n, z_2^n)$ given $u^n$
- Superimpose $x^n$
Coding scheme

Split rate $R = R_0 + R_1 + R_2 + R_3$

Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$
Summary – Two disturbance constraints

➢ Use Marton coding and superposition coding

Transmitted:

\[ u^n \xleftarrow{z_1^n} x^n \]

\[ Z_1 \xrightarrow{f_1} Z_2 \]

\[ Z_2 \xrightarrow{f_2} Z_2 \]

Layered codebook

\[ u^n \xrightarrow{z_1^n} x^n \]

\[ u^n \xrightarrow{z_1^n} z_2^n \]

\[ u^n \xrightarrow{z_1^n} z_1^n \]

\[ u^n \xrightarrow{z_2^n} z_2^n \]

\[ u^n \xrightarrow{z_2^n} x^n \]

Observed structure:

Superposition

Superposition
Summary – Transmitter-centric aspect

Disturbance-constrained communication

- Link to interference channels (single constraint case recovers Han-Kobayashi)
- Two constraints: Layered coding scheme
Putting the pieces together — 3-DIC

BC aspect

MAC aspect
Putting the pieces together — 3-DIC

**BC aspect**

Marton coding and superposition coding

**MAC aspect**

Interference decoding

+
Putting the pieces together — 3-DIC

- Codebooks as in *disturbance-constrained communication*
  Fixed distributions $p(u_k, x_k)$ for all users $k$
Putting the pieces together — 3-DIC

- Codebooks as in **disturbance-constrained communication**
  Fixed distributions $p(u_k, x_k)$ for all users $k$
Putting the pieces together — 3-DIC

- Codebooks as in **disturbance-constrained communication**
  - Fixed distributions $p(u_k, x_k)$ for all users $k$

![Diagram of 3-pair interference channels]

- Receivers use **interference decoding**
  - Use full knowledge of interfering codebooks
  - Non-unique simultaneous decoding
3-DIC capacity region inner bound

**Theorem**

\[ \mathcal{R} = \text{FM} \left\{ \bigcup_{p} \mathcal{R}_1(p) \cap \mathcal{R}_2(p) \cap \mathcal{R}_3(p) \right\}, \]

where \( p \) is of the form \( p = p(q)p(u_1, x_1|q)p(u_2, x_2|q)p(u_3, x_3|q) \), is an inner bound to the capacity region of the 3-DIC.

Where

- FM is a specialized Fourier–Motzkin elimination
- \( \mathcal{R}_1, \mathcal{R}_2, \) and \( \mathcal{R}_3 \) are rate regions in \( \mathbb{R}^{18} \)
3-DIC capacity region inner bound

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Details are quite involved. But easy to evaluate by computer.
Theorem

\[ R = \text{FM} \left\{ \bigcup_p R_1(p) \cap R_2(p) \cap R_3(p) \right\}, \]

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Where

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**Remarks:**
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Remarks:

- Regions \( \mathcal{R}_k \) are not convex (due to min terms)
- FM (Fourier–Motzkin) cannot be evaluated symbolically
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Remarks:

- Strictly contains interference-decoding inner bound
  - Interference as noise < Interference decoding < This inner bound
3-DIC capacity region inner bound

**Theorem**

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**Remarks:**

- Strictly contains interference-decoding inner bound
  - Interference as noise \(<\) Interference decoding \(<\) This inner bound

- Achieves 2-pair marginal capacities
  - Subsumes Han-Kobayashi scheme for every 2-pair subchannel
Example

![Diagram of 3-pair interference channels]

- \(X_1\) with \(\{0, 1, 2\}\)
- \(X_2\) with \(\{0, 1, 2\}\)
- \(X_3\) with \(\{0, 1, 2\}\)

Outputs:
- \(Y_1\) with \(\{0, \ldots, 4\}\)
- \(Y_2\) with \(\{0, \ldots, 4\}\)
- \(Y_3\) with \(\{0, \ldots, 4\}\)

Weights:
- \(0 \rightarrow 0\)
- \(2 \rightarrow 0\)
- \(1 \rightarrow 1\)

Constraints:
- \(\{0, 1\}\)
Example: Rate regions

- Interference as noise
- Interference decoding
- Layered scheme & Interference Decoding
Example: Rate regions

Interference as noise

Interference decoding

Layered scheme & Interference Decoding
Example: Rate regions
Summary – Achievable rate region for 3-DIC

- New inner bound to the capacity region
  - Non-symbolic Fourier–Motzkin elimination
  - Numerous modes of saturation
Extension to channels with noise

(to be presented at ISIT 2012)
Extension to channels with noise

(to be presented at ISIT 2012)

- Combined interference $S_k$ passes through a noisy channel $S_k \rightarrow S_k'$
  - Arbitrary DMC $p(s'_k|s_k)$
- $h_k$ and $f_k$ remain injective in each argument
Extension to channels with noise

(to be presented at ISIT 2012)

This channel:
- Has structure similar to 3-DIC
- Contains the Gaussian IC as a special case
Inner bound for channels with noise

Theorem

\[ \mathcal{R} = \text{FM} \left\{ \bigcup_{p} \mathcal{R}_1(p) \cap \mathcal{R}_2(p) \cap \mathcal{R}_3(p) \right\}, \]

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As before,

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  Details are still quite involved . . .
  
  But still easy to evaluate by computer
Summary – Channels with noise

- Inner bound techniques continue to work
- Achievable regimes subsumes Han–Kobayashi scheme
Conclusion

▶ Receiver-centric aspect

- Interference decoding: Exploit structure of combined interference
- Simultaneous non-unique decoding is optimal
- Subsumes treating interference as noise
Conclusion

- **Receiver-centric aspect**
  - Interference decoding: Exploit structure of combined interference
  - Simultaneous non-unique decoding is optimal
  - Subsumes treating interference as noise

- **Transmitter-centric aspect**
  - Introduced communication with disturbance constraints
  - Found rate–disturbance region for single constraint
  - Connection to interference channels
  - Inner bound for deterministic case with two constraints
Conclusion

► Receiver-centric aspect
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  – Subsumes treating interference as noise

► Transmitter-centric aspect
  – Introduced communication with disturbance constraints
  – Found rate–disturbance region for single constraint
  – Connection to interference channels
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► Combination of the two aspects
  – New capacity inner bound for 3-DIC: improves previous bounds
  – Transmission scheme extends to noisy channels
Thanks!
Inner bound on the rate–disturbance region

**Theorem**

The set of \((R, R_{d,1}, R_{d,2}) \in \mathbb{R}^3_+\) satisfying

\[
\begin{align*}
    R &\leq H(Y) \\
    R_{d,1} + R_{d,2} &\geq I(Z_1; Z_2 | U) \\
    R - R_{d,1} &\leq H(Y | Z_1, U) \\
    R - R_{d,2} &\leq H(Y | Z_2, U) \\
    R - R_{d,1} - R_{d,2} &\leq H(Y | Z_1, Z_2, U) - I(Z_1; Z_2 | U) \\
    2R - R_{d,1} - R_{d,2} &\leq H(Y | Z_1, Z_2, U) + H(Y | U) - I(Z_1; Z_2 | U)
\end{align*}
\]

for some joint distribution \(p(u, x)\), is an inner bound to the rate–disturbance region

(This is optimal in some cases, e.g., if \(Y = X\))
Coding scheme

Split rate $R = R_0 + R_1 + R_2 + R_3$
Let $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$

Decoding conditions:

\[ R_3 < H(Y \mid Z_1, Z_2, U) \]
\[ \tilde{R}_1 + R_3 < H(Y \mid Z_2, U) + I(Z_1; Z_2 \mid U) \]
\[ \tilde{R}_2 + R_3 < H(Y \mid Z_1, U) + I(Z_1; Z_2 \mid U) \]
\[ \tilde{R}_1 + \tilde{R}_2 + R_3 < H(Y \mid U) + I(Z_1; Z_2 \mid U) \]
\[ R_0 + \tilde{R}_1 + \tilde{R}_2 + R_3 < H(Y) + I(Z_1; Z_2 \mid U) \]

(plus some encoding conditions)
3-DIC inner bound — Conditions for $R_1$ (first receiver)

**Conditions:** Encoder conditions, and

$$r_1 + \min\{r_{21} + r_{31}, H(S_1 | c_{21}, c_{31})\} < H(Y_1 | c_1, c_{21}, c_{31}) + t_1$$
3-DIC inner bound — Conditions for $R_1$ (first receiver)

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$$r_1 + \min\{r_{21} + r_{31}, H(S_1 | c_{21}, c_{31})\} < H(Y_1 | c_1, c_{21}, c_{31}) + t_1$$

From disturbance-constrained communication

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From interference decoding

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3-DIC inner bound — Conditions for $\mathcal{R}_1$ (first receiver)

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$R_{20} \xrightarrow{u_2^n} x_{21}^n \xrightarrow{} \tilde{R}_{21}$

$R_{30} \xrightarrow{u_3^n} x_{31}^n \xrightarrow{} \tilde{R}_{31}$
3-DIC inner bound — Conditions for $\mathcal{R}_1$ (first receiver)

**Conditions:** Encoder conditions, and

$$r_1 + \min\{r_{21} + r_{31}, H(S_1 | c_{21}, c_{31})\} < H(Y_1 | c_1, c_{21}, c_{31}) + t_1$$

**From disturbance-constrained communication**

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<td>$R_{30}$</td>
<td>$u_3^n \rightarrow x_{31}^n$</td>
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3-DIC inner bound — Conditions for $\mathcal{R}_1$ (first receiver)

Condition (example):

$$\tilde{R}_{13} + R_{11} + \min \{ R_{20} + \tilde{R}_{21} + \tilde{R}_{31},$$

$$R_{20} + \tilde{R}_{21} + H(X_{31} | U_3, Q),$$

$$R_{20} + \tilde{R}_{31} + H(X_{21} | U_2, Q),$$

$$R_{20} + H(X_{21} | U_2, Q) + H(X_{31} | U_3, Q),$$

$$\tilde{R}_{31} + H(X_{21} | Q),$$

$$H(S_1 | U_3, Q) \leq H(Y_1 | U_1, X_{12}, U_3, Q)$$

$$+ I(X_{12}, X_{13} | U_1, Q)$$
3-DIC inner bound — Conditions for $\mathcal{R}_1$ (first receiver)

**Condition (example):**

$$
\tilde{R}_{13} + R_{11} + \min \left\{ R_{20} + \tilde{R}_{21} + \tilde{R}_{31}, \right.
\begin{align*}
R_{20} + \tilde{R}_{21} + H(X_{31} | U_3, Q), \\
R_{20} + \tilde{R}_{31} + H(X_{21} | U_2, Q), \\
R_{20} + H(X_{21} | U_2, Q) + H(X_{31} | U_3, Q), \\
\tilde{R}_{31} + H(X_{21} | Q), \\
H(S_1 | U_3, Q) \right\} \leq H(Y_1 | U_1, X_{12}, U_3, Q) \\
+ I(X_{12}; X_{13} | U_1, Q)
\end{align*}
$$

- 45 conditions of this type, plus 5 encoder conditions
- Not hard to evaluate by computer
Example conditions for $\tilde{R}_1$ (first receiver)

\[ \tilde{R}_{13} + R_{11} + \min \{ R_{20} + \tilde{R}_{21} + \tilde{R}_{31}, \]
\[ R_{20} + \tilde{R}_{21} + I(S_1; S'_1 | X_2, U_3, Q), \]
\[ R_{20} + \tilde{R}_{31} + I(S_1; S'_1 | U_2, X_3, Q), \]
\[ R_{20} + I(S_1; S'_1 | U_2, U_3, Q), \]
\[ \tilde{R}_{31} + I(S_1; S'_1 | X_3, Q), \]
\[ I(S_1; S'_1 | U_3, Q) \leq I(X_1, X_2, X_3; Y_1 | U_1, X_{12}, U_3, Q) \]
\[ \quad + I(X_{12}; X_{13} | U_1, Q) \]

- Saturation interpretation still holds
- Conditional noisy channel capacities replace typical set size
Example conditions for $R_1$ (first receiver)

$$\tilde{R}_{13} + R_{11} + \min \{ R_{20} + \tilde{R}_{21} + \tilde{R}_{31},$$
$$R_{20} + \tilde{R}_{21} + H(X_{31} | U_3, Q),$$
$$R_{20} + \tilde{R}_{31} + H(X_{21} | U_2, Q),$$
$$R_{20} + H(X_{21} | U_2, Q) + H(X_{31} | U_3, Q),$$
$$\tilde{R}_{31} + H(X_{21} | Q),$$
$$H(S_1 | U_3, Q) \} \leq H(Y_1 | U_1, X_{12}, U_3, Q)$$
$$+ I(X_{12}; X_{13} | U_1, Q)$$

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- Conditional noisy channel capacities replace typical set size