Revisiting the inner and outer bounds for the two receiver broadcast channel

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Outline of talk

- An observation and a thought experiment
- Existing bounds
- A comparison between them
- A different way of thinking
- What is missing...
- More examples
Some preliminaries

Recall: Superposition coding can be used to achieve the union of rate pairs \((R_1, R_2)\) satisfying

\[
R_1 \leq I(U; Y_1)
\]
\[
R_1 + R_2 \leq I(U; Y_1) + I(X; Y_2 | U)
\]
\[
R_1 + R_2 \leq I(X; Y_2)
\]

over all \(p(u, x)\).

Korner-Marton and El Gamal established that the union of rate pairs \((R_1, R_2)\) satisfying

\[
R_1 \leq I(U; Y_1)
\]
\[
R_1 + R_2 \leq I(U; Y_1) + I(X; Y_2 | U)
\]
\[
R_2 \leq I(X; Y_2)
\]

over all \(p(u, x)\) forms an outer bound to the capacity region.
Observation 1: The above inner and outer bounds seem great for a degraded scenario (where $Y_1$ is the weaker receiver).

Observation 2: All the capacity regions are established by showing that these two regions coincide.

Question: Are the two regions (inner and outer bounds) the same or are they different?
Observation 1: The above inner and outer bounds seem great for a degraded scenario (where $Y_1$ is the weaker receiver).

Observation 2: All the capacity regions are established by showing that these two regions coincide.

Question: Are the two regions (inner and outer bounds) the same or are the different?

Using Observation 1, a natural antipodal setting seems to be when there is no degradedness in the picture.
A non-degradable broadcast channel is one where there does not exist a non-trivial decomposition of the form

\[ X \rightarrow \tilde{X} \rightarrow Y_1, Y_2 \]

If

- \( P : X \leftrightarrow Y_1 \)
- \( Q : X \leftrightarrow Y_2 \)

then there does not exist \( M \), a non-trivial \(|X| \times |X|\) stochastic matrix such that

- \( P = P_1 \times M \)
- \( Q = Q_1 \times M \)
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The simplest example of a non-degradable broadcast channel is the BSSC
Non-degradable BC

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The simplest example of a non-degradable broadcast channel is the BSSC

Thus intuitively, BSSC is a perfect channel to compare the bounds
The binary skew-symmetric broadcast channel (BSSC)

Figure: Binary Skew Symmetric Channel
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- A different way of thinking
- What is missing...
- More examples
An achievable region (Marton ’79)

Most of this talk, assume \( R_0 = 0 \) (no common message)

Recall that the following rates are achievable

\[
R_1 \leq I(U, W; Y_1) \\
R_2 \leq I(V, W; Y_2) \\
R_1 + R_2 \leq \min \{ I(W; Y_1), I(W; Y_2) \} + I(U; Y_1 | W) \\
+ I(V; Y_2 | W) - I(U; V | W)
\]
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$$+ I(V; Y_2|W) - I(U; V|W)$$

This is the best achievable region known to-date

- Not even a special carefully constructed channel where one can beat this
- Obviously, no proof of optimality
Paper: Capacity of a class of broadcast channels (more capable)

The union of rate pairs \((R_1, R_2)\) satisfying

\[
R_1 \leq I(U; Y_1) \\
R_2 \leq I(V; Y_2) \\
R_1 + R_2 \leq I(U; Y_1) + I(X; Y_2 | U) \\
R_1 + R_2 \leq I(V; Y_2) + I(X; Y_1 | V)
\]

over all \(p(u, v, x)\) constitutes an outer bound.
Outer bound: El Gamal (Asilomar ’76, IT ’79)

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\]

over all \(p(u, v, x)\) constitutes an outer bound.

Remark: Because this bound was not explicitly stated, this was not well-known (registered)

Call this bound the UV-OB.
Körner-Márton (IT ’79)

Paper: An achievable rate region for 2-receiver discrete memoryless broadcast channels (Márton)

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over all \(p(v, x)\) constitutes an outer bound.

Remark: This was used in establishing the capacity of the semi-deterministic broadcast channel
Körner-Márton outer bound

Let $\mathcal{R}_a$ be the union of rate pairs $(R_1, R_2)$ satisfying

\begin{align*}
R_1 &\leq I(X; Y_1) \\
R_2 &\leq I(V; Y_2) \\
R_1 + R_2 &\leq I(V; Y_2) + I(X; Y_1|V)
\end{align*}

over all $p(v, x)$.

Let $\mathcal{R}_b$ be the union of rate pairs $(R_1, R_2)$ satisfying

\begin{align*}
R_1 &\leq I(U; Y_1) \\
R_2 &\leq I(X; Y_2) \\
R_1 + R_2 &\leq I(U; Y_1) + I(X; Y_2|U)
\end{align*}

over all $p(v, x)$.
Körner-Márton outer bound

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over all $p(v, x)$.

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over all $p(v, x)$.

The region $\mathcal{R}_a \cap \mathcal{R}_b$ became known as the Körner-Márton outer bound.
The following comparisons are immediate:

- $\text{UV-OB} \subseteq \text{KM-OB}$
- $\text{UV-OB} \subseteq \text{Sato's outer bound}$
Remarks

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Further KM-OB matches the capacity region in all special cases where capacity was established.
The union of rate pairs \((R_1, R_2)\) satisfying

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R_1 \leq I(U, W; Y_1)
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R_2 \leq I(V, W; Y_2)
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over all \(p(u)p(v)p(w, x|u, v)\) constitutes an outer bound.
The union of rate pairs \((R_1, R_2)\) satisfying

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Also showed that

- this bound \(\subseteq\) UV-OB \(\subset\) KM-OB
- BSSC: UV-OB \(\subset\) KM-OB (Surprise (😊))
The union of rate pairs \((R_1, R_2)\) satisfying

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Also showed that

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- BSSC: UV-OB \(\subset\) KM-OB (Surprise \(\smiley\))

However Nair-Wang ('08) showed that the above bound \(\equiv\) UV-OB
Comparing inner and outer bounds

- KM-OB = Marton Inner Bound (MIB) in all special cases where capacity was established.
- UV-OB ⊂ KM-OB
  - This implies that KM-OB ≠ UV-OB
Comparing inner and outer bounds

- KM-OB = Marton Inner Bound (MIB) in all special cases where capacity was established.

- UV-OB $\subset$ KM-OB
  - This implies that KM-OB $\neq$ UV-OB

- Is it true that $UV - OB = MIB$?

To answer this, we again look at the BSSC.
**Conjecture:** [N-Wang ’08] For all \((U, V) \rightarrow X \rightarrow (Y_1, Y_2)\)

\[
I(U; Y_1) + I(V; Y_2) - I(U; V) \leq \max\{I(X; Y_1), I(X; Y_2)\}
\]

If the conjecture is true

- Maximum \(R_1 + R_2\) achievable by Märton’s strategy is 0.3616..
- Maximum \(R_1 + R_2\) contained in the outer bound is 0.3725.. (N-EG ’07)
- Thus inner and outer bound regions differ (!)
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The conjecture is true when \(P(X = 0) \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1]\)

Recall: No cardinality bounds on auxiliary random variables
BSSC: Comparing the bounds

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\[ I(U; Y_1) + I(V; Y_2) - I(U; V) \leq \max\{I(X; Y_1), I(X; Y_2)\} \]

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Recall: No cardinality bounds on auxiliary random variables

[Gohari-Anantharam ’09]
- Proved: sufficient to consider \(|U| \leq |X|, |V| \leq |X|, X = f(U, V)\) to establish conjecture
- Proved: inner and outer bounds differ for BSSC
Sum-rate bounds for BSSC

Extending the perturbation method [Jog-Nair ’09] established the conjecture, i.e.

For all $p(u, v, x)$, s/t $(U, V) \rightarrow X \rightarrow (Y_1, Y_2)$

$$I(U; Y_1) + I(V; Y_2) - I(U; V) \leq \max\{I(X; Y_1), I(X; Y_2)\}$$

This implies that sum-rate bounds of BSSC are:
- Marton’s inner bound: 0.3616...
- UV-OB: 0.37255...
- KM-OB: 0.3743...
Sum-rate bounds for BSSC

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- Marton’s inner bound: 0.3616...
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- KM-OB: 0.3743...

Aside: Generalizing the arguments of [Jog-Nair ’09], it is known that for all \( p(u, v, x, y_1, y_2) \), s/t \( (U, V) \rightarrow X \rightarrow (Y_1, Y_2) \)

\[
I(U; Y_1) + I(V; Y_2) - I(U; V) \leq \max\{I(X; Y_1), I(X; Y_2)\}
\]
as long as \(|X| = 2\)
Open question

What is the optimal sum-rate of BSSC.

Answering this will determine whether:
  - Which bound/s are loose

Possibly require new ideas
Other bounds

- Liang, Liang-Kramer had concurrently developed similar outer bounds
  - Not known if they were better than existing bounds (e.g., KM-OB)

- Liang, Kramer, Shamai developed the **New-Jersey outer bound** (’08)

- Nair developed another outer bound (’08). **No-sum-rate outer bound**

The following relations were established:

- No sum-rate outer bound $\subseteq$ New-Jersey outer bound
- New-Jersey outer bound $\subseteq \left( \text{outer bound (Nair-El Gamal)} \cap \text{outer-bound(Liang, Liang-Kramer)} \right)$

Remark: Equivalences or strict inclusions are not established
New-Jersey outer bound (LKS ’08)

The union of rate triples \((R_0, R_1, R_2)\) satisfying

\[
\begin{align*}
R_0 &\leq \min\{I(T; Y|W_1), I(T; Z|W_2)\} \\
R_1 &\leq I(U; Y|W_1) \\
R_2 &\leq I(V; Z|W_2) \\
R_0 + R_1 &\leq I(T, U; Y|W_1) \\
R_0 + R_1 &\leq I(U; Y|T, W_1, W_2) + I(T, W_1; Z|W_2) \\
R_0 + R_2 &\leq I(T, V; Z|W_2) \\
R_0 + R_2 &\leq I(V; Z|T, W_1, W_2) + I(T, W_2; Y|W_1) \\
R_0 + R_1 + R_2 &\leq I(U; Y|T, V, W_1, W_2) + I(T, V, W_1; Z|W_2) \\
R_0 + R_1 + R_2 &\leq I(V; Z|T, U, W_1, W_2) + I(T, U, W_2; Y|W_1) \\
R_0 + R_1 + R_2 &\leq I(U; Y|T, V, W_1, W_2) + I(T, W_1, W_2; Y) + I(V; Z|T, W_1, W_2) \\
R_0 + R_1 + R_2 &\leq I(V; Z|T, U, W_1, W_2) + I(T, W_1, W_2; Z) + I(U; Y|T, W_1, W_2)
\end{align*}
\]

for some \(p(u)p(v)p(t)p(w_1, w_2|u, v, t)p(x|u, v, t, w_1, w_2)p(y, z|x)\) constitutes an outer bound.
An equivalent evaluable region

The union of rate triples \((R_0, R_1, R_2)\) satisfying

\[
R_0 \leq \min\{I(W; Y), I(W; Z)\}
\]

\[
R_0 + R_1 \leq I(U; Y|W) + \min\{I(W; Y), I(W; Z)\}
\]

\[
R_0 + R_2 \leq I(V; Z|W) + \min\{I(W; Y), I(W; Z)\}
\]

\[
R_0 + R_1 + R_2 \leq \min\{I(W; Y), I(W; Z)\} + I(U; Y|W) + I(X; Z|U, W)
\]

\[
R_0 + R_1 + R_2 \leq \min\{I(W; Y), I(W; Z)\} + I(V; Z|W) + I(X; Y|V, W)
\]

for some \(p(u, v, w)p(y, z|x)\) is equivalent to the NJ-outer bound.

Proof idea: same as Nair-Wang (’08)

- Suffices to consider \(|W| \leq |X| + 7; |U|, |V| \leq |X| + 2\)
- If one is interested in sumrate
  - suffices to consider \(|U|, |V| \leq |X|; W = \emptyset\).
- When \(R_0 = 0\) this region is \(\equiv UV\)-OB
Thus when $R_0 = 0$ we have the following current situation:

- no sum-rate outer bound $\subseteq$ UV-OB
  - No $W$ required for the outer bound (!)
  - For inner bound, we know that $W$ is critical even when $R_0 = 0$.  

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Broadcast Channel  
Mar 9, 2010  
21 / 32
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What about no sum-rate outer bound?

How does the sum-rate of BSSC compare?

Ans: It is at least 0.37251...
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What about **no sum-rate outer bound**?

How does the sum-rate of BSSC compare?

Ans: It is at least 0.37251...

Belief: no sum-rate outer bound $\equiv$ UV-OB
All the above outer bounds are basically algebraic manipulations that

- Start from Fano’s inequality
- Use Data processing inequality
- Use Csiszár sum lemma
- Identify auxiliary random variables in terms of $M_1, M_2, Y^{i-1}, Z_i^n, etc$
Reflections

All the above outer bounds are basically algebraic manipulations that

- Start from Fano’s inequality
- Use Data processing inequality
- Use Csiszár sum lemma
- Identify auxiliary random variables in terms of $M_1, M_2, Y^{i-1}, Z^n_{i+1}, etc$

Remark: Irrespective of the algebra we do not seem to beat the UV-OB using above approach.

Hence, start from a clean slate.
Reflections

All the above outer bounds are basically algebraic manipulations that
- Start from Fano’s inequality
- Use Data processing inequality
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Remark: Irrespective of the algebra we do not seem to beat the UV-OB using above approach.

Hence, start from a clean slate.

Borrows ideas and results from Images of a set by Körner-Márton (’77)
Outline of talk

- An observation and a thought experiment
- Existing outer bounds
- A comparison between them
- A different way of thinking
- What is missing...
- More examples
Images of a set ...

Given \( p(x) \), consider \( B \subset T_{\varepsilon}^{(n)}(X^n) \)

Image(B) w.r.t channel \( X \mapsto Y \) is

\[
\inf \frac{1}{n} \log P(C) : C \subseteq T_{\varepsilon}^{(n)}(Y^n), P(y^n \in C| x^n) > 1 - \varepsilon, \forall x^n \in B
\]
Images of a set ...

Given $p(x)$, consider $\mathcal{B} \subset \mathcal{T}_\epsilon^{(n)}(X^n)$

Image$(\mathcal{B})$ w.r.t channel $X \mapsto Y$ is

- $\inf \frac{1}{n} \log P(C) : C \subseteq \mathcal{T}_\epsilon^{(n)}(Y^n), P(y^n \in C|x^n) > 1 - \epsilon, \forall x^n \in \mathcal{B}$

Remarks

- If $|B| = 1$ then $|C^*| \approx 2^{nH(Y|X)}$, and Image$(\mathcal{B}) = -I(X; Y)$
- If $\mathcal{B}$ is a code book of size $2^{nR}$, then Image$(\mathcal{B}) = R - I(X; Y)$
- If $\mathcal{B} \neq \emptyset$, then $-I(X; Y) \leq \text{Image}(\mathcal{B}) \leq 0$
Images of a set ... 

Given $p(x)$, consider $\mathcal{B} \subset \mathcal{T}_\epsilon^{(n)}(\mathcal{X}^n)$ 

Image($\mathcal{B}$) w.r.t channel $X \mapsto Y$ is

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- If $\mathcal{B}$ is a code book of size $2^{nR}$, then Image($\mathcal{B}$) = $R - I(X; Y)$
- If $\mathcal{B} \neq \emptyset$, then $-I(X; Y) \leq$ Image($\mathcal{B}$) $\leq 0$

Theorem (KM-77)

If Image($\mathcal{B}$)$_{X \mapsto Y}$ $\geq t$, then Image($\mathcal{B}$)$_{X \mapsto Z}$ $\geq T_{Y \mapsto Z}(t)$, where

$$T_{Y \mapsto Z}(t) = \min \{ r - I(U; Z) : r - I(U; Y) \geq t, 0 \leq r \leq I(U; Y) \}$$
A reasoning

Consider a good code book (maximal error probability is small) (Willems ’91)

Let \( B_i = \{ x^n(i, j), j \in (1, ..., 2^{nR_2}) \} \).

Properties

1. Each \( B_i \) is a \( 2^{nR_2} \) code book for receiver \( Z \)
   - Image \((B_i)_{X \rightarrow Z} \geq R_2 - I(X; Z)\)
   - Therefore, Image \((B_i)_{X \rightarrow Y} \geq T_{Z \rightarrow Y}(R_2 - I(X; Z))\)
Consider a good code book (maximal error probability is small) (Willems ’91)

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2. The receiver $Y$ can distinguish between $\mathcal{B}_i$, i.e. Images $(\mathcal{B}_i)_{X \rightarrow Y}$ are disjoint
   - Therefore $R_1 + T_{Z \rightarrow Y}(R_2 - I(X; Z)) \leq 0$
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2. The receiver $Y$ can distinguish between $B_i$, i.e. Images $(B_i)_{X \rightarrow Y}$ are disjoint
   - Therefore $R_1 + T_{Z \rightarrow Y} (R_2 - I(X; Z)) \leq 0$

Thus any good codebook must satisfy

$$R_1 + T_{Z \rightarrow Y} (R_2 - I(X; Z)) \leq 0$$

$$R_2 + T_{Y \rightarrow Y} (R_1 - I(X; Y)) \leq 0$$ (interchange roles)
How good is the outer bound (OB)

\[ R_1 + T_{Z \rightarrow Y}(R_2 - I(X; Z)) \leq 0 \]
\[ R_2 + T_{Y \rightarrow Z}(R_1 - I(X; Y)) \leq 0 \]

Remarks:
- OB ⊆ UV-OB
Comparison

How good is the outer bound (OB)

\[ R_1 + T_{Z \rightarrow Y}(R_2 - I(X; Z)) \leq 0 \]
\[ R_2 + T_{Y \rightarrow Z}(R_1 - I(X; Y)) \leq 0 \]

Remarks:
- OB ⊆ UV-OB
- **Litmus test:** Sumrate of BSSC
  - Sumrate of OB (BSSC) = 0.37255.. = Sumrate of UV-OB (BSSC)
  - Fails the litmus test ☹️
Comparison

How good is the outer bound (OB)

\[ R_1 + T_{Z \rightarrow Y} \left( R_2 - I(X; Z) \right) \leq 0 \]
\[ R_2 + T_{Y \rightarrow Z} \left( R_1 - I(X; Y) \right) \leq 0 \]

Remarks:

- OB \subseteq UV-OB
- **Litmus test**: Sumrate of BSSC
  - Sumrate of OB (BSSC) = 0.37255.. = Sumrate of UV-OB (BSSC)
  - Fails the litmus test 😊

**Silver lining**: There is another property that a good code book must have
Figure: An overcounting
Remarks

- We figured a possible over counting with OB
- Do we need to bother about this overlap (over-counting)
  - No - degraded, less noisy, more capable (superposition coding)
    - Disjoint images in weaker receiver can be made to be disjoint in stronger receiver (without losing anything in exponent)
  - No - semideterministic
    - The images on the deterministic receiver are point sets (!)
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**Surprise**: These are precisely the classes where capacity is known (!)

Therefore one needs to show either of the two:

- We need not bother with this over-counting
- This over-counting does matter and UV-OB can be made tighter.
Remarks

Looked at existing bounds

- OB (with $R_0$) is a simple *evaluatable* region
- when $R_0 = 0$, UV-OB still rules!

Introduced Litmus test 😊

- Compare the sum rate to that of BSSC
Remarks

Looked at existing bounds

- OB (with $R_0$) is a simple evaluatable region
- when $R_0 = 0$, UV-OB still rules!

Introduced Litmus test 😊
  - Compare the sum rate to that of BSSC

Derived a new looking bound using a much more intuitive reasoning

- Showed that it is as good as UV-OB
- However litmus test failed
- Identified a possible over counting (weakness in outer bound)
Outline of talk

- Existing outer bounds
- A comparison between them
- A different way of thinking
- What is missing...
- More examples
BISO broadcast channels

BISO: (Binary-Input Symmetric-Output)
A channel is BISO if the channel transition matrix satisfies

\[ P(Y = k|X = 0) = P(Y = -k|X = 1), \forall k \]

Examples: BSC, BEC
BISO broadcast channels

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[Geng-Nair-Shamai-Wang ’10]

Consider a BC where \( X \leftrightarrow Y_1, X \leftrightarrow Y_2 \) are BISO channels

Then the following are equivalent:

- Neither is more capable than the other, i.e. \( \exists p_1, p_2 \) s.t
  \[ I(X; Y_1) > I(X; Y_2)|_{P(X=0)=p_1}, \quad I(X; Y_1) < I(X; Y_2)|_{P(X=0)=p_2}. \]

- Marton’s inner bound \( \subset \) UV-OB

There are BISO broadcast channels with \(|Y| \geq 4\) which are not more-capable comparable
Thank You

More on Thursday