On the Construction and Decoding of Cyclic LDPC Codes

Chao Chen

Joint work with Prof. Baoming Bai from Xidian University

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Cyclic LDPC codes form an important class of structured LDPC codes.

- As cyclic codes, they can be simply encoded with shift register.
- As LDPC codes, they provide good performance under iterative decoding with a reasonable decoding complexity.
- They have relatively large minimum distance.
- Some codes can be transformed into quasi-cyclic (QC) codes through row and column permutations on the parity-check matrix.
Some Known Constructions

- Construction based on finite geometries (Lin et al.)
- Construction based on idempotents (Shibuya et al., Tomlinson et al.)
- Construction based on matrix decomposition (Lin et al.)
1. Introduction

Code Features

- The defining parity-check matrix is a circulant matrix or a column of circulant matrices.
- The parity-check matrix is highly redundant.
- The column weight $\gamma$ is relatively large.
- The corresponding Tanner graph is free of length-4 cycles.
- The minimum distance of the code is at least $\gamma + 1$. (Massey bound)
Idempotent

**Definition:** Let $R_n$ be the ring of residue classes of $F_q[x]$ modulo $x^n - 1$. Then a polynomial $e(x)$ of $R_n$ is called an *idempotent* if $e^2(x) = e(x)$.

**Properties:**

- For a cyclic code $C$, there exists a unique idempotent $e(x)$ that generates $C$, called the generating idempotent of $C$.
- Let $e^\perp(x)$ be the generating idempotent of $C^\perp$, the dual code of $C$, then $e^\perp(x) = 1 - x^n e(x^{-1})$. 
Modular Golomb Ruler

**Definition:** A set of integers \( \{a_i : 0 \leq a_i < n, 1 \leq i \leq \gamma\} \) is called a *Golomb ruler modulo* \( n \) *with* \( \gamma \) *marks*, if the differences \((a_i - a_j) \mod n\) are distinct for all ordered pairs \((i, j)\) with \(i \neq j\).

An example:

![Golomb ruler modulo 31 with 6 marks](image)

**Figure:** A Golomb ruler modulo 31 with 6 marks.

An inherent constraint: \( \gamma \times (\gamma - 1) \leq (n - 1) \).
Three Algebraic Constructions of Modular Golomb Rulers

- Singer construction (projective geometry plane)
- Bose construction (Euclidean geometry plane)
- Ruzza construction
Code Definition

Consider a cyclic LDPC code $C$ of length $n$ over $F_q$, whose parity-check matrix is an $n \times n$ circulant

$$H = \begin{bmatrix}
c_0 & c_1 & \cdots & c_{n-1} \\
c_{n-1} & c_0 & \cdots & c_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & \cdots & c_0
\end{bmatrix}.$$  

The above code is specified by the polynomial

$$c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} = c_{a_1} x^{a_1} + c_{a_2} x^{a_2} \cdots + c_{a_\gamma} x^{a_\gamma},$$

where $\gamma$ is the row weight of $H$ and $c_{a_i} \neq 0$ ($i = 1, \cdots, \gamma$).
Main Results

- If $c(x)$ is an idempotent, then the minimum distance of $C$ satisfies

$$d_{min} \leq \begin{cases} 
\gamma + 1, & \text{if } c_0 = 0; \\
\gamma - 1, & \text{if } c_0 = 1; \\
\gamma, & \text{otherwise.}
\end{cases}$$

- The Tanner graph corresponding to $H$ is free of length-4 cycles if and only if \{a_1, a_2, \cdots, a_\gamma\} is a Golomb ruler modulo $n$ with $\gamma$ marks.

- According to Massey bound, if the Tanner graph is free of length-4 cycles, then $d_{min} \geq \gamma + 1$.

- If $c(x)$ is an idempotent and \{a_1, a_2, \cdots, a_\gamma\} is a modular Golomb ruler, then the minimum distance of $C$ is exactly $\gamma + 1$. 
Let $\alpha$ be a primitive element of $F_q$, then $0, \alpha^0, \cdots, \alpha^{q-2}$ give all elements of $F_q$. Let $\beta \in F_q$, then it can be mapped to a $(q-1) \times (q-1)$ binary matrix $A(\beta)$, as shown below.

The matrix $A(\beta)$ is called the *matrix dispersion* of $\beta$ over $F_2$. 

\[
0 \rightarrow A(0) = \begin{bmatrix}
0
\end{bmatrix}
\quad \quad \quad \quad \quad \\
\alpha^i \rightarrow A(\alpha^i) = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]
Procedure for Finite Field based Construction

First construct an \( m \times n \) matrix over \( F_q \), called the base matrix.

\[
W = \begin{bmatrix}
w_0 \\
w_1 \\
\vdots \\
w_{m-1}
\end{bmatrix} = \begin{bmatrix}
w_{0,0} & w_{0,1} & \cdots & w_{0,n-1} \\
w_{1,0} & w_{1,1} & \cdots & w_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
w_{m-1,0} & w_{m-1,1} & \cdots & w_{m-1,n-1}
\end{bmatrix}.
\]

Then replace each entry of \( W \) by its matrix dispersion and form the following matrix as the parity-check matrix

\[
H(W) = \begin{bmatrix}
A(w_{0,0}) & A(w_{0,1}) & \cdots & A(w_{0,n-1}) \\
A(w_{1,0}) & A(w_{1,1}) & \cdots & A(w_{1,n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
A(w_{m-1,0}) & A(w_{m-1,1}) & \cdots & A(w_{m-1,n-1})
\end{bmatrix}.
\]
Design Constraint on $\mathbf{W}$

For $0 \leq i, j \leq m - 1$, $i \neq j$ and $0 \leq h, k \leq q - 2$,

$$d(\alpha^h \mathbf{w}_i, \alpha^k \mathbf{w}_j) \geq n - 1,$$

where $d$ denotes the Hamming distance.

The constraint is called the $\alpha$-multiplied row distance (RD) constraint, which guarantees that the Tanner graph corresponding to $\mathbf{H}(\mathbf{W})$ is free of length-4 cycles.
Pseudo-Cyclic Code and MDS Code

**Definition:** A linear block code of length $n$ is a pseudo-cyclic code with parameter $\beta \in F_q$, if for any codeword $(c_0, c_1, \cdots, c_{n-1})$, its pseudo-cyclic $(\beta c_{n-1}, c_0, \cdots, c_{n-2})$ also forms a codeword.

**Definition:** An $(n, k)$ linear block code is a maximum-distance-separable (MDS) code, if the minimum distance $d_{min} = n - k + 1$. 
Code Construction

Consider the following $n \times n$ matrix over $F_q$

$$W = \begin{bmatrix}
 w_0 & w_1 & \cdots & w_{n-2} & w_{n-1} \\
 \alpha w_{n-1} & w_0 & \cdots & w_{n-3} & w_{n-2} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 \alpha w_2 & \alpha w_3 & \cdots & w_0 & w_1 \\
 \alpha w_1 & \alpha w_2 & \cdots & \alpha w_{n-1} & w_0
\end{bmatrix},$$

where the rows are codewords of a $(n, 2)$ pseudo-cyclic MDS code with $\beta = \alpha$.

It can be proved that the $W$ satisfies the $\alpha$-multiplied RD constraint.

Through matrix dispersion, the obtained $H(W)$ defines a QC-LDPC code.
Main Result

From the above QC-LDPC code, a cyclic LDPC code can be obtained by transforming $H(W)$ to a circulant parity-check matrix through row and column permutations.
Automorphism Group of a Code

**Definition:** Let $C$ be a binary linear block code of length $n$. The set of coordinate permutations that map $C$ to itself forms a group under composition operation. The group is called the automorphism group of $C$, denoted by $Aut(C)$.

For a binary cyclic code of odd length $n$, the automorphism group contains the following two cyclic subgroups:

- $S_0$: The set of permutations $\tau^0, \tau^1, \ldots, \tau^{n-1}$, where
  \[
  \tau^k : j \rightarrow (j + k) \mod n.
  \]

- $S_1$: The set of permutations $\zeta^0, \zeta^1, \ldots, \zeta^{m-1}$, where
  \[
  \zeta^k : j \rightarrow (2^k \cdot j) \mod n,
  \]
  and $m$ is the smallest positive integer such that $2^m \equiv 1 \mod n$. 
Properties of $Aut(C)$

- Let $C^\perp$ be the dual code of $C$, then $Aut(C^\perp) = Aut(C)$.
- Let $\pi \in Aut(C)$. If $H$ is a parity-check matrix of $C$, then $\pi H$ also forms a parity-check matrix of $C$. 
Decoder Diversity

- Two party-check matrices are called *non-equivalent* if they cannot be obtained from each other only by row permutations.
- The basic idea is to construct multiple non-equivalent parity-check matrices based on $\text{Aut}(C)$. Different decoding attempts can be made on these parity-check matrices, thus providing decoder diversity gain.
Main Results

- For cyclic LDPC codes constructed from idempotents and modular Golomb rulers, $S_0$ and $S_1$ cannot be used to generate non-equivalent parity-check matrices.

- For cyclic LDPC codes constructed from pseudo-cyclic MDS codes with two information symbols, $S_1$ can be used to generate non-equivalent parity-check matrices.
Simulation Results

A (341, 160) cyclic LDPC code is constructed from (31, 2) pseudo-cyclic MDS code over $F_{32}$. The BPSK modulation over AWGN channel is assumed. The maximum number of iterations is set to be 100.

![Graph showing FER vs. Eb/N0]
References


References


Thanks!