Group Secret Key Agreement
over
State-Dependent
Wireless Broadcast Channels

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Institute of Network Coding, CUHK, Hong Kong
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Motivation

- Consider *m trusted terminals* that communicate through a wireless channel
- **Goal:** Creating a *common secret key K*, which is concealed from a *passive eavesdropper Eve*
Motivation

- **Current Approach:** Using public-key cryptography; Based on:
  - Some *unproven* hardness problems
  - The computational power of Eve is limited
Motivation

• **Alternative Approach:** Propose a scheme that guarantees information theoretical secrecy

• **Benefits:**
  • It is the strongest notion of secrecy
  • No matter how computationally powerful Eve is, she cannot find any information about the secret key

• **Disclaimer! (use it at your own risk!)**
  • not claiming that this approach is a replacement for the current cryptographic systems
Motivation

• **Wireless Networks:**
  
  • **Disadvantage:** Eavesdropping on wireless networks is much easier than wired network
  
  • **Advantages:** The channels from the source to different destinations are different and are changing over time
  
  • **Main idea:** Use the non-uniformity nature (fluctuations) of the wireless medium
Problem Statement

- **Goal**: $m$ trusted (authenticated) terminals aim to create a common secret key which will be secret from a passive eavesdropper Eve.

- There is a broadcast channel from one of the terminals (Alice) to the others including Eve.

- Trusted terminals have access to a costless public channel.

- Terminals can interact in many rounds.

- In general, the exact characterization of the secrecy rate is unknown!
**Problem Statement**

**Wireless Channel Models**

- **Different Broadcast Models:**
  1. We assume that the wireless broadcast channel acts as a broadcast packets erasure channel
  2. We approximately model different SNR levels by using a deterministic model
  3. We investigate a state-dependent Gaussian broadcast channel

- **Assumption:** The channels from Alice to the rest of terminal are independent, namely:

\[
P_{X_1\ldots X_m X_E | X_A}(x_1, \ldots, x_m, x_E | x_A) = P_{X_E | X_A}(x_E | x_A) \prod_{i=1}^{m} P_{X_i | X_A}(x_i | x_A)
\]
Previous Results
Previous Results

Wiretap Channel (Wyner 1975, Csiszar and Korner 1978)

- **Goal:** Alice wants to send a message to Bob over a broadcast channel where Eve overhears.

\[
\mathcal{P}[\hat{W} = W] > 1 - \epsilon \quad \text{and} \quad \frac{1}{n} I(W; Z^n) < \epsilon
\]

- If Eve’s channel is “less noisy” than Bob’s => \( C_s = 0 \)

\[
C_s = \max_{U-X-Y,Z} \left[ I(U; Y) - I(U; Z) \right]
\]
Previous Results
Feedback Can Help (Maurer 1993)

• The same setup as wiretap channel
• A rate-unlimited costless public channel is available
• Even if Eve’s channel is “less noisy” than Bob’s, we may have:
  \[ C_s > 0 \]
Previous Results
Multi-terminal Secret Key Sharing Problem (Csiszar and Narayan 2008)

- **Assumptions:** A broadcast channel and a public channel is available; Terminal 0 broadcasts; Eve has only access to public channel; Terminals can interact in many rounds

\[
S(X_0; \cdots; X_{m-1}) = \max_{P_{X_0}} \left[ H(X_0, \ldots, X_{m-1}) - \max_{\lambda \in \Lambda} \sum_{B \subseteq [0:m-1]} \lambda_B H(X_B | X_{B^c}) \right]
\]

- **\( R_{CO} \)** is the smallest rate of public discussion \( F \) such that \( X_{[0:,m-1]}^n \) is recoverable from \( (X_i^n, F) \)
Previous Results
Multi-terminal Secret Key Sharing Problem with Side Information

- **Assumptions:** Similar to the previous problem; Eve has access to public channel + side information

- The problem is still open even for two terminals
- A corollary of the previous result (but no achievability proposed by Csiszar & Narayan):

\[
S(X_0; \cdots; X_{m-1} \| Z) \leq \max_{P_{X_0}} \left[ H(X_0, \ldots, X_{m-1} \| Z) - \max_{\lambda \in \Lambda} \sum_{B \subseteq \{0:m-1\}} \lambda_B H(X_B \| X_{B^c}, Z) \right]
\]
Previous Results

Extensions

• Multi-terminal Secret Key Sharing Problem with Side Information (Gohari and Anantharam 2010)
  • The same setup as before
  • Upper and lower bounds for the secret key generation (the achievability is hard to evaluate; infinite aux. rv.s)

• (Csiszar and Narayan 2013) and (Chan and Zheng 2014)
  • Extension to multi-input multi-output channel but without eavesdropper side information
  • Upper and lower bounds for the secret key generation
Upper Bound
Multi-terminal Secret Key Sharing Problem with Side Information

• By [CsiszarNarayan08] and adding a dummy terminal, we have (but no achievability proposed by C&N):

\[
S(X_0; \ldots; X_{m-1}||Z) \leq \max_{P_{X_0}} \left[ H(X_0, \ldots, X_{m-1}|Z) - \max_{\lambda \in \Lambda} \sum_{B \subseteq [0:m-1]} \lambda_B H(X_B|X_{B^c}, Z) \right]
\]

• If the channels are independent, we can further simplify:

\[
S(X_0; \ldots; X_{m-1}||Z) \leq \max_{P_{X_0}} \min_{i \in [1:m-1]} I(X_0; X_i|Z)
\]

\[
\leq \min_{i \in [1:m-1]} \max_{P_{X_0}} I(X_0; X_i|Z)
\]

![Diagram of Broadcast Channel with Alice, Bob, and Eve connected to X0 through X1 to Xm-1 and Z]
Erasure Broadcast Channel
Erasure Broadcast Channel

- The wireless channel is modelled by a packet erasure channel
- Each terminal either receives packets sent by Alice or not
- Channels are independent
- The input and output symbols are packets of length $L$ from $\mathbb{F}_q$
Erasure Broadcast Channel

- **Question:** What is the secret key sharing capacity in this setup?

- **Theorem:** The capacity of this problem is

  \[ S(X_0; \ldots ; X_{m-1} \parallel Z) = (1 - \delta)\delta_E \times (L \log_2 q) \]

  packet length in bits

- **The result does not depend on m!**
Sketch of the Achievability

Private Phase

- Alice sends $n$ packets $\{x_1, \ldots, x_n\}$
- Bob and Calvin receive $(1 - \delta)n$ packets each
- Eve observes $(1 - \delta_E)(1 - \delta)n$ packets from each of these sets
- $\Rightarrow$ There exist some packets that Bob (Calvin) receives but Eve does not

$$|N_B| \approx |N_C| \approx \delta(1 - \delta)n$$
$$|N_{BC}| \approx (1 - \delta)^2n$$
$$|N_{B\setminus E}| \approx |N_{C\setminus E}| \approx \delta(1 - \delta)\delta_E n$$
$$|N_{BC\setminus E}| \approx (1 - \delta)^2\delta_E n$$
Sketch of the Achievability
Public Discussion (Initial Phase)

• Bob and Calvin send back the indices of their packets publicly
• Alice reproduce $\mathcal{N}_B$, $\mathcal{N}_C$, and $\mathcal{N}_{BC}$
• If a genie tells Alice the indices of Eve’s packets we are done => The green packets form a key
• Question: What we can do?

$$|\mathcal{N}_B| \approx |\mathcal{N}_C| \approx \delta(1 - \delta)n$$

$$|\mathcal{N}_{BC}| \approx (1 - \delta)^2n$$

$$|\mathcal{N}_{B\setminus E}| \approx |\mathcal{N}_{C\setminus E}| \approx \delta(1 - \delta)\delta_{E}n$$

$$|\mathcal{N}_{BC\setminus E}| \approx (1 - \delta)^2\delta_{E}n$$
Sketch of the Achievability
Public Discussion (Initial Phase)

• **Lemma:** It is possible to create the same number as of green sets, linear combinations out of $\mathcal{N}_B$, $\mathcal{N}_C$ and over $\mathcal{N}_{BC}$ so that these packets are secure from Eve.

• Alice sends the coefficients of these new green linear combinations publicly, Eve does not gain any information $\Rightarrow$ A set of keys: $K_B$, $K_C$, and $K_{BC}$

$$|\mathcal{N}_B| \approx |\mathcal{N}_C| \approx \delta(1-\delta)n$$

$$|\mathcal{N}_{BC}| \approx (1-\delta)^2n$$

$$|\mathcal{N}_{B\setminus E}| \approx |\mathcal{N}_{C\setminus E}| \approx \delta(1-\delta)\delta_E n$$

$$|\mathcal{N}_{BC\setminus E}| \approx (1-\delta)^2\delta_E n$$
Sketch of the Achievability
Public Discussion (Reconciliation Phase)

• $K_{BC}$ can be part of the final key

• Using $K_B$ and $K_C$, Alice can share a new key with Bob and Calvin over the public channel

• So in total, the final key size is:

$$|K_B| + |K_{BC}| = |N_{B\setminus E}| + |N_{BC\setminus E}| \approx (1 - \delta)\delta n$$

• In general, Alice can use a network code to reconcile the key over the public channel
Erasure Broadcast Channel

Shortcomings of modelling a wireless channel with an erasure channel

• A packet is declared as erased if some number of bits have been corrupted
  => Eve can exploit the remaining bits

• The actual channel is a continuous channel with varying SNR
  => Need a more sophisticated model to capture the different SNR levels
Deterministic Broadcast Channel
Deterministic Broadcast Channel

• The wireless channel is modelled by a deterministic channel*

• There are $s + 1$ channel states modelling different SNR levels

\[ X_r[t] = F_{S_r[t]} X_0[t] \]

• Channels are independent

• Assume CSI at receivers

Deterministic Broadcast Channel

Received vector at r'th terminal
\[ X_r[t] = F_{s_r[t]} X_0[t] \]

Channel State \( \in [0 : s] \)

\[ 0 = \ker F_s \subset \ker F_{s-1} \subset \cdots \subset \ker F_0 = \mathbb{F}_q^L \]

\[ \text{rank}(F_i - F_{i-1}) = \text{rank}(F_i) - \text{rank}(F_{i-1}) \]

\[ F_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ F_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
Sketch of the Achievability
Superposition Coding

- We can find subspaces \( \Pi_1, \ldots, \Pi_s \) such that \( \Pi_i \cap \Pi_j = 0 \) and

\[
\begin{align*}
\Pi_1 \oplus \ker F_1 &= \mathbb{F}_q^L \\
\Pi_2 \oplus \Pi_1 \oplus \ker F_2 &= \mathbb{F}_q^L \\
& \vdots \\
\Pi_s \oplus \cdots \oplus \Pi_1 \oplus \ker F_s &= \mathbb{F}_q^L
\end{align*}
\]

- Alice uses superposition coding:

\[
X_0[t] = X_{0,1}[t] + \cdots + X_{0,s}[t] \quad \text{where} \quad X_{0,i} \in \Pi_i
\]

- Vector \( X_{0,i}[t] \) is received by the \( r \)'th terminal only if \( S_r \geq i \)

\[\Rightarrow \text{we have } s \text{ independent erasure channels!}\]

- \( X_{0,i}[t] \) is received with erasure probability \( \theta_i \triangleq \sum_{j=0}^{i-1} \delta_i \)
Sketch of the Achievability

Superposition Coding

• We can find subspaces $\Pi_1, \ldots, \Pi_s$ such that $\Pi_i \cap \Pi_j = \emptyset$ and

  \[ \pi_1 \oplus \ker F_1 = \mathbb{F}_q^L \]
  \[ \pi_2 \oplus \pi_1 \oplus \ker F_2 = \mathbb{F}_q^L \]
  \[ \vdots \]
  \[ \pi_s \oplus \cdots \oplus \pi_1 \oplus \ker F_s = \mathbb{F}_q^L \]

• Alice uses superposition coding:

  \[ X_0[t] = X_{0,1}[t] + \cdots + X_{0,s}[t] \]
  \[ \text{where } \quad X_{0,i} \in \pi_i \]

• Vector $X_{0,i}[t]$ is received by the r'th terminal only if $S_r \geq i$
  \[ \Rightarrow \text{ we have } s \text{ independent erasure channels!} \]

• $X_{0,i}[t]$ is received with erasure probability

  \[ \theta_i \triangleq \sum_{j=0}^{i-1} \delta_i \]
Theorem: The secret key generation capacity for the deterministic broadcast channel is:

\[ S(X_0; \ldots; X_m \mid Z) = \sum_{j=1}^{s} \theta_j (1 - \theta_j) \left[ \text{rank } F_j - \text{rank } F_{j-1} \right] \log_2 q \]

- \( \theta_j \): Erasure probability of the \( j \)th layer.
- \( \text{rank } F_j \): Number of messages in the \( j \)th layer (in bits).
- \( \log_2 q \): Information capacity.
Gaussian Broadcast Channel
Gaussian Broadcast Model

- There is a **Gaussian broadcast channel** from Alice to other terminals
- Channels are **independent**
- There are $s + 1$ **channel states** having different SNR levels
- Assume CSI at receivers

$$X_r[t] = h_{Sr}[t]X_0[t] + Z_r[t]$$
Gaussian Broadcast Model

- There is a Gaussian broadcast channel from Alice to other terminals
- Channels are independent
- There are $s + 1$ channel states having different SNR levels
- Assume CSI at receivers

\[
X_r[t] = h_{S_r[t]} X_0[t] + Z_r[t]
\]

- Channel gain $\in \mathbb{R}$
- $h_0 \leq \cdots \leq h_s$
- Channel State $\in [0 : s]$
Upper Bound
Gaussian Broadcast Channel

• **Theorem:** (By combining [Csiszar-Narayan-08] and [Chan-Zheng-14] and independence of channels):

The secret key generation capacity of the Gaussian broadcast channel using public discussion is upper bounded as follows:

\[
C_s \leq \sup_{P_{X_0}} \frac{1}{L} \mathbb{E}[||X_0||^2] \leq P \min_{j \in [1:m]} I(X_0; X_j | Z)
\]

\[
\leq \frac{1}{2} L \sum_{i=0}^{s} \sum_{j=0}^{s} \delta_i \delta_j \log \left( 1 + \frac{h_i^2 P}{1 + h_j^2 P} \right)
\]
Theorem: (By combining [Csiszar-Narayan-08] and [Chan-Zheng-14] and independence of channels):
The secret key generation capacity of the Gaussian broadcast channel using public discussion is upper bounded as follows:

\[
C_s \leq \sup_{P_{X_0}} \min_{j \in [1:m]} \frac{1}{L} \mathbb{E}[[||X_0||^2] \leq P} \sum_{i=0}^{s} \sum_{j=0}^{s} \delta_i \delta_j \log \left( 1 + \frac{h_i^2 P} {1 + h_j^2 P} \right)
\]

State probability \( \delta_i \delta_j \)
Channel gain \( h_0 \leq \cdots \leq h_s \)
Input power budget

\( C_s \leq \)
Sketch of the Achievability

• We want to mimic the orthogonality operation of the deterministic channel

• By using a properly designed layered wiretap code:
  • => we can introduce orthogonal layers (each layer acts as an erasure channel)

• On each layer, we apply the interactive scheme devised for the erasure channel
Nested Message Set, Degraded Wiretap Channel

Alice

\( W_1, \ldots, W_s \rightarrow \text{Enc} \)

\( X^n_A \rightarrow h_0X_A + Z_0 \rightarrow \text{Dec} \)

\( h_1X_A + Z_1 \rightarrow \text{Dec} \rightarrow \hat{W}_1 \)

\( h_sX_A + Z_s \rightarrow \text{Dec} \rightarrow \hat{W}_1, \ldots, \hat{W}_s \)

Channel gain \( \in \mathbb{R} \)

\( h_0 \leq \cdots \leq h_s \)

• Code Design Goals:
  • Message \( W_i \) should be decodable by receivers \( Y_i, \ldots, Y_s \)
  • All receivers \( Y_0, \ldots, Y_{i-1} \) should be ignorant about message \( W_i \)
  • Now, we have the orthogonality operation among messages \( W_i \)
Nested Message Set, Degraded Wiretap Channel

- Alice uses **superposition coding**: \( X_A[t] = X_{A,1}[t] + \cdots + X_{A,s}[t] \)
- She maps \( W_i \) to \( X_{A,i} \) as follows:
  - Construct codebook \( \hat{C}_i(2^{LR_i}, L) \) by choosing independent symbols from \( \mathcal{N}(0, P_i) \) where:
    \[
    \hat{R}_i = \frac{1}{2} \log \left( 1 + \frac{h_i^2 P_i}{1 + h_i^2 I_i} \right)
    \]
  - Each codebook \( \hat{C}_i \) is divided into \( 2^{LR_i} \) bins where:
    \[
    R_i = \frac{1}{2} \left[ \log \left( 1 + \frac{h_i^2 P_i}{1 + h_i^2 I_i} \right) - \log \left( 1 + \frac{h_{i-1}^2 P_i}{1 + h_{i-1}^2 I_i} \right) \right]
    \]
  - Message \( W_i \) is mapped to the bin index and \( X_{A,i} \) is a random sequence from that bin
Nested Message Set, Degraded Wiretap Channel

- Alice uses superposition coding: \( X_A[t] = X_{A,1}[t] + \cdots + X_{A,s}[t] \)
- She maps \( W_i \) to \( X_{A,i} \) as follows:
  - Construct codebook \( \hat{C}_i(2^{L\hat{R}_i}, L) \) by choosing independent symbols from \( \mathcal{N}(0, P_i) \) where:
    \[
    \hat{R}_i = \frac{1}{2} \log \left( 1 + \frac{h_i^2 P_i}{1 + h_i^2 I_i} \right)
    \]
  - Each codebook \( \hat{C}_i \) is divided into \( 2^{L\hat{R}_i} \) bins where:
    \[
    R_i = \frac{1}{2} \left[ \log \left( 1 + \frac{h_i^2 P_i}{1 + h_i^2 I_i} \right) - \log \left( 1 + \frac{h_{i-1}^2 P_i}{1 + h_{i-1}^2 I_i} \right) \right]
    \]
  - Message \( W_i \) is mapped to the bin index and \( X_{A,i} \) is a random sequence from that bin
Nested Message Set, Degraded Wiretap Channel

- Alice uses superposition coding: $X_A[t] = X_{A,1}[t] + \cdots + X_{A,s}[t]$.

- She maps $W_i$ to $X_{A,i}$ as follows:
  - Construct codebook $\hat{C}_i(2^{LR_i}, L)$ by choosing independent symbols from $\mathcal{N}(0, P_i)$ where:
    $$\hat{R}_i = \frac{1}{2} \log \left( 1 + \frac{h_i^2 P_i}{1 + h_i^2 I_i} \right)$$
    $$I_i = \sum_{j=i+1}^{s} P_j$$
    $$\hat{C}_i$$ is divided into $2^{LR_i}$ bins where:
    $$R_i = \frac{1}{2} \left[ \log \left( 1 + \frac{h_i^2 P_i}{1 + h_i^2 I_i} \right) - \log \left( 1 + \frac{h_{i-1}^2 P_i}{1 + h_{i-1}^2 I_i} \right) \right]$$
  - Message $W_i$ is mapped to the bin index and $X_{A,i}$ is a random sequence from that bin.
Sketch of the Achievability, cont.

• The $r$’th receiver with channel state $S_r = i$:
  • can **successively decode** messages up to layer $i$
  • is **ignorant** about messages of layers above $i$

• $\implies$ Each $W_i$ experiences an independent erasure channel with erasure probability:

\[
\theta_i \triangleq \sum_{j=0}^{i-1} \delta_i
\]

• For each layer, **run the interactive scheme for erasure channels**

• The achievable secret key generation rate, for each power allocation $\{P_i\}$ is:

\[
R_s = \sum_{i=1}^{s} \theta_i (1 - \theta_i) R_i
\]
Power Optimization Problem
Sketch of the Achievability

• The maximum achievable secrecy rate is given by:

\[
R_s = \left\{ \begin{array}{l}
\max \quad \sum_{i=1}^{s} \theta_i (1 - \theta_i) R_i \\
\text{subject to} \quad \sum_{i=1}^{s} P_i \leq P \\
\quad P_i \geq 0, \quad \forall i \in [1 : s].
\end{array} \right.
\]

• This is a not a convex optimization problem!

• Constraints are affine => KKT equations give necessary conditions
  • All of KKT solutions can be found by a backtracking algorithm
  • ==> The optimum solution can be found!

• The upper and lower bounds do not match!
Results: Asymptotic Behaviour

• Assuming high-dynamic range, i.e., \( h_i \gg h_{i-1} \) and high-SNR regime:

• The upper and lower bounds match in a degrees of freedom sense (\( h_i = Q^{\gamma_i} \)):

\[
\text{DoF}_s \triangleq \lim_{Q \to \infty} \frac{C_s}{\frac{1}{2} \log Q} = L \sum_{i=1}^{s} (\gamma_i - \gamma_{i-1})(1 - \theta_i)\theta_i
\]
The achievable rate and the upper bound as a function of $h_1$ with $P$: (a) $P=0.01$, (b) $P=0.1$, (c) $P=1$, and (d) $P=10$, in a setup with 3 equiprobable states ($h_0 = -5\text{dB}$, $-5\text{dB} < h_1 < 30 \text{ dB}$, and $h_2 = 30\text{dB}$).
The achievable rate and the upper bound as a function of $g_1$ and $g_2$ with $P=10$ in a setup with 4 equiprobable states ($h_0 = -5\text{dB}$, $h_1 = \min[g_1,g_2]$, $h_2 = \max[g_1,g_2]$, and $h_3 = 30\text{dB}$).
Fraction of total power $P$ allocated to each layer by the proposed scheme for $P = 0.1, 1.0, 10, 100$ in a setup consisting 36 equiprobable states ($h_0 = -5$dB, $h_1 = -4$dB, ..., $h_{35} = 30$dB).
Challenges

• For a usual cryptographic system:
  An attack can be done by an adversary who has very high computational power

• In the proposed setup:
  An attack can be done by an adversary who has multiple antennas at many different places

• In general, it is hard to estimate the Eve’s channel statistics
Thank You!