Network Coding for Line Networks with Broadcast Channels

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Network Communication

- Network: represent by a graph
- Example: network of 3 networks
  - **Line**, e.g., an ethernet bus
    - Single-path routing common for simplicity, control, security
  - **Star**, e.g., a radio network cell
    - Hub controls and monitors
    - Scalable, can isolate failures
  - **Ring**, e.g., an optical network
    - Two paths protect against failures
1) Models

- **Wireline**: edge capacity constraints $C_{u,v}$ for edge $(u,v)$

![Diagram](image)

- **Nodes** can also be bottlenecks, e.g., processor energy, speed, bus bandwidth constraints. Capacity $C_u$ for node $u$
Wireless with Broadcast Channels (BCs)

- **Broadcast constraint** via $X_u$ rather than $X_{u,u-1}$ and $X_{u,u+1}$
- **Time-frequency slots:** no interference
- **General:** add interference via $Y_u$ rather than $Y_{u-1,u}$ and $Y_{u+1,u}$

\[
Y_{1,2} = f_{1,2}(X_1, Z_1) \\
Y_{3,2} = f_{3,2}(X_3, Z_3) \\
Y_{3,4} = f_{3,4}(X_3, Z_3)
\]

\[
Y_{2,1} = f_{2,1}(X_2, Z_2) \\
Y_{2,3} = f_{2,3}(X_2, Z_2) \\
Y_{4,3} = f_{4,3}(X_4, Z_4)
\]
Traffic Sessions

- Traffic Sessions: $u \rightarrow D(u) = \{v(1), \ldots, v(L)\}$, rate $R(u \rightarrow D(u))$
  - **Unicast**: up to $n(n-1)$ sessions between node-pairs
  - **Broadcast**: $n$ sessions (one node to all other nodes)
  - **Multicast**: $n(2^{n-1}-1)$ sessions (one node $u$ to a node set $D(u)$)

- Node constraints: can place sources & sinks at different sub-nodes for different problems

```
Traffic Sessions

[Graph showing traffic sessions between nodes]
```

```c
s_{1 \rightarrow \{2,3,4\}}
```

```c
C_{1,2} S_{2 \rightarrow 4} C_{3,2} C_{3,4} S_{4 \rightarrow \{1,2\}}
```

```c
1o 2i 3o 4i
```

```c
1i 2o 3i 4o
```

```c
C_{1}  C_{2}  C_{3}  C_{4}
```

```c
C_{2,1}  C_{2,3}  C_{4,3}
```

```c
C_{2,1}  C_{2,3}  C_{4,3}
```

```c
C_{2,1}  C_{2,3}  C_{4,3}
```
2) Wireline: How to Communicate?

- Network coding helps: many “butterflies”
- Guess: Routing, copying, and “butterfly” binary linear network coding is optimal. For equal-length packets:

\[ \begin{align*}
&1o \rightarrow [W_{1→3}] \\
&1i \rightarrow [W_{1→3}, W_{4→1}] \\
&2i \rightarrow [W_{2→4} \oplus W_{4→1}, W_{3→2}] \\
&3i \rightarrow [W_{2→4} \oplus W_{4→1}, W_{1→3}, W_{3→2}] \\
&4i \rightarrow [W_{2→4}, W_{4→1}] \\
&1o \rightarrow [W_{1→3}] \\
&2i \rightarrow [W_{1→3}, W_{4→1}] \\
&3i \rightarrow [W_{2→4} \oplus W_{4→1}, W_{2→4}] \\
&4i \rightarrow [W_{4→1}] \\
&2o \rightarrow [W_{1→3} \oplus W_{4→1}] \\
&3o \rightarrow [W_{2→4} \oplus W_{4→1}, W_{3→2}] \\
&4o \rightarrow [W_{2→4}, W_{4→1}] \\
&3o \rightarrow [W_{2→4} \oplus W_{4→1}] \\
&4o \rightarrow [W_{4→1}] \\
\end{align*} \]
Non-uniform Packet Lengths and Rates

Let:

\[
A \oplus B = \begin{cases} 
[A_1 \oplus B_1, \ldots, A_m \oplus B_m] & m \leq n \\
[A_1 \oplus B_1, \ldots, A_n \oplus B_n, A_{n+1}, \ldots, A_m] & m > n 
\end{cases}
\]

\[
A \otimes B = \begin{cases} 
[A_1 \oplus B_1, \ldots, A_m \oplus B_m, B_{m+1}, \ldots, B_n] & m \leq n \\
[A_1 \oplus B_1, \ldots, A_n \oplus B_n, A_{n+1}, \ldots, A_m] & m > n 
\end{cases}
\]
Notes

- Method seems simple but requires careful control.
  Each node u treats **8 sets** of messages differently
  1) Left-to-right (LR) messages through node u
  2) Right-to-left (RL) messages through node u
  3) Left-to-right (LRu) messages also destined for u
  4) Right-to-left (RLu) messages also destined for u
  5) L-to-R and R-to-L messages “stopping” at node u (u)
  6) Node u messages going to left and right (u,LR)
  7) Node u messages going to right (u,R)
  8) Node u messages going to left (u,L)

- **Converse:**
  - Classic cut bounds insufficient
  - **Progressive edge-cut** bounds give the capacity
    (and include classic cut bounds)
3) Progressive Edge Cuts (Kramer-Savari '06)

- Consider a general edge set $E$ and session set $S$
- Initialize: remove (1) edges in $E$; (2) edges of sources not in $S$; (3) edges out of nodes directed-sense\(^1\) disconnected from $S$
- Repeat: test if an $s$ in $S$ is undirected-sense\(^2\) disconnected from any of its sinks. If so, remove $s$ and then edges out of nodes directed-sense\(^1\) disconnected from the remaining sources.
- Successful removal of all sources: $\sum_{k \in S} R_k \leq \sum_{e \in E} C_e$
- Example: $E=\{(1,3),(2,3)\}$ and $S=\{s_1,s_2\}$: $R_1+R_2 \leq 2$ if $C_e=1$ for all $e$

\(^1\)based on functional dependencies
\(^2\)based on fd-separation in functional dependence graphs (Kramer '98)
Line Network Rate Constraints

- **Edges:** get basic routing rates (cf. classic cut-set bound)
  \[ \sum_{i=1..u} \sum_{D(i) \text{ with a node in } \{u+1..n\}} R(i \rightarrow D(i)) \leq C_{u,u+1} \quad (L \rightarrow R) \]
  \[ \sum_{i=u..n} \sum_{D(i) \text{ with a node in } \{1..u-1\}} R(i \rightarrow D(i)) \leq C_{u,u-1} \quad (R \rightarrow L) \]

- **Nodes:** node u incoming and outgoing rates plus max(L→R rates, R→L rates) (see graph on p. 8):
  \[ \sum_{D(u)} R(u \rightarrow D(u)) + \sum_{v} \sum_{\text{Traffic stops at } u} R(v \rightarrow D(v)) \]
  \[ + \max \left( \sum_{i=1..u-1} \sum_{D(i) \text{ with a node in } \{u+1..n\}} R(i \rightarrow D(i)), \right. \]
  \[ \sum_{i=u+1..n} \sum_{D(i) \text{ with a node in } \{1..u-1\}} R(i \rightarrow D(i)) \right) \leq C_{u} \]
Application of Edge Cuts to Lines

- Bound for node u: choose $E=\{(u_i,u_o)\}$ and $S=\{L\to R \text{ sources across } u\} \cup \{u \text{ incoming and outgoing sources}\}$

- Example: $u=3$ with $E=\{(3_i,3_o)\}$
  - Remove $(3_i,3_o)$; $s$ right of node 3 and $s$ left of node 3 having sinks left of node 3 only; edges right of $u$ and $(3_o,2_i)$
  - Can remove all sources **including** node 3 outgoing sources
  - Gives new (non-classic) $L\to R$ bounds; similarly get new $R\to L$ bounds; these bounds, combined with the classic cut bounds, define the multiple-multicast capacity region

![Diagram of network with edge cuts](image)
4) What about Wireless?

- **Problem:** capacity of BCs with feedback is unknown
- **Partial resolution:** capacity is known for some cases
  - orthogonal channels
  - deterministic channels
  - physically degraded channels, including physically degraded Gaussian BCs [El Gamal, 1978]
- Do the coding/converse methods extend to our networks?
- **Answer:** yes! See our paper “Network coding for line networks with broadcast channels,” Entropy, vol. 14, 2012
- Paper gives a general achievable region, and converses for the above cases and for packet erasure channels
Notes: the **Progressive Edge Cut Tool**

- Includes classic cuts as special cases
- Applies to network coding (a classic edge-cut bound does **not**)
- Generalizes naturally to **wireless** networks to include any coding
- **Wireless** example below: using \( E=\{(1,3),(2,3)\} \) and \( S=\{1,2\} \) gives

\[
R_1 \leq \min[I(X_1;Y_2Y_3|X_2X_3), I(X_1X_2;Y_3|X_3)]
\]
\[
R_2 \leq \min[I(X_3;Y_1|X_1), I(X_2;Y_3|X_1X_3)]
\]
\[
R_1 + R_2 \leq I(X_1X_2;Y_3|X_3)
\]

\( \{ \text{Classic cuts} \} \)

← **Progressive cut**
Summary

Line Networks:
- even wireline problems require careful coding and have sophisticated capacity regions;
- ideas extend to certain broadcasting scenarios;
- for general BCs: we first need the capacities of BCs with (generalized) feedback;
- including interference will be even tougher!
Extra Slides
**Classic Cut-Set Bound**

- Partition **nodes** into two sets $N$ and $N^C$
- Let $S$ be the set of sessions originating in $N$ with a sink in $N^C$
- Cut $E$ is the set of edges starting in $N$ and ending in $N^C$
- Classic cut bound: $\sum_{k \in S} R_k \leq \sum_{e \in E} C_e$
- Example: ring with 2 unicasting sessions and unit-edge capacities. We have: $R_1 \leq 2$, $R_2 \leq 1$
Line Networks with Edge Constraints Only

- **Routing**: bounds for \((u,u+1)\) and \((u+1,u)\):
  
  \[
  \sum_{i=1..u} \sum_{D(i) \text{ with a node in } \{u+1..n\}} R(i \rightarrow D(i)) \leq C_{u,u+1} \quad (L \rightarrow R)
  \]
  
  \[
  \sum_{i=u+1..n} \sum_{D(i) \text{ with a node in } \{1..u\}} R(i \rightarrow D(i)) \leq C_{u+1,u} \quad (R \rightarrow L)
  \]

- **Classic cut-set bound**
  - For cut \(\{(u,u+1)\}\) is just \((L \rightarrow R)\)
  - For cut \(\{(u+1,u)\}\) is just \((R \rightarrow L)\)

- So routing (+ copying for multicast) is rate-optimal
fd-Separation (Kramer ‘98)

Let $A$, $B$ and $C$ be vectors whose entries are RVs (vertices) of a FDG.

Success after the following implies $I(A ; B | C) = 0$ (cf. Pearl 1988)

- Consider only vertices and edges met when moving backward from the vertices in $A$, $B$, or $C$ (“causality”)
- Remove the outgoing edges of vertices disconnected from the sources in a directed sense
- Check if there is no undirected path from “$A$” to “$B$”

Ex: $I(W_1 ; \hat{W}_1 | Y_{2,3} Y_{1,3} Z_{3,1}) = 0$

$I(W_2 ; \hat{W}_2 | Y_{2,3} Y_{1,3} Z_{3,1} W_1) = 0$