Chapter 6
Strong Typicality

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The Chinese University of Hong Kong
6.1 Strong AEP
Setup

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- \mathbf{X} = (X_1, X_2, \cdots, X_n). \text{ Then }

\[ p(\mathbf{X}) = p(X_1)p(X_2)\cdots p(X_n). \]
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p(X) = p(X_1)p(X_2)\cdots p(X_n).
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- Assume \( |\mathcal{X}| < \infty \).
- Let the base of the logarithm be 2, i.e., \( H(X) \) is in bits.
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• Consider $x \in \mathcal{X}^n$. 
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- Let $N(x; x)$ be the number of occurrences of $x$ in the sequence $x$. 

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Let $x = (1, 3, 2, 1, 1)$. 

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The empirical distribution of $x$ is $\frac{3}{5}$, $\frac{1}{5}$, $\frac{1}{5}$. 
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- $n^{-1}N(x; \mathbf{x})$ is the relative frequency of $x$ in $\mathbf{x}$.
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- The empirical distribution of $x$ is $\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\}$. 
Definition 6.1 The strongly typical set $T_{[X]_{\delta}}^n$ with respect to $p(x)$ is the set of sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ such that

$$N(x; \mathbf{x}) = 0 \text{ for } x \notin S_X$$

and

$$\sum_x \left| \frac{1}{n}N(x; \mathbf{x}) - p(x) \right| \leq \delta,$$

where $\delta$ is an arbitrarily small positive real number. The sequences in $T_{[X]_{\delta}}^n$ are called strongly $\delta$-typical sequences.

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• In other words, $n^{-1}N(x; x) \approx p(x)$ for all $x \in \mathcal{X}$.

• Therefore, if $x$ is strongly typical, the empirical distribution of $x$ is approximately equal to the generic distribution $p(x)$.

• If $x$ is strongly typical, then $p(x_k) > 0$ for all $k$ because of (1).
**Theorem 6.2 (Strong AEP)**  There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x^2 \in T_n[X]$, then
   $$n\left(H(X) + \eta\right) \leq p(x) \leq n\left(H(X) + \eta\right).$$

2) For $n$ sufficiently large,
   $$\Pr\{X^2 \in T_n[X]\} > 1.$$
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

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1) If $x \in T_{[X,\delta]}^n$, then

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2) For $n$ sufficiently large,

$$\Pr\{X \in T_{[X,\delta]}^n\} > 1 - \delta.$$

3) For $n$ sufficiently large,

$$(1 - \delta)2^{n(H(X)-\eta)} \leq |T_{[X,\delta]}^n| \leq 2^{n(H(X)+\eta)}.$$
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Proof Idea

- If $x$ is strongly typical, then the empirical distribution is “about right”.

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- If the empirical distribution is about right, then everything else, including the empirical entropy, would be about right, i.e.,

$$-\frac{1}{n} \log p(x) \approx H(X).$$
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- If the empirical distribution is about right, then everything else, including the empirical entropy, would be about right, i.e.,

$$-\frac{1}{n} \log p(x) \approx H(X).$$

- This is equivalent to $p(x) \approx 2^{-nH(X)}$. 
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1) If $x \in T^n[X]^{\delta}$, then

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Proof
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X] \delta}$, then

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1. To prove Property 1, for $x \in T^n_{[X] \delta}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$
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because $N(x;x) = 0$ for all $x \not\in S_X$. Then
**Theorem 6.2 (Strong AEP)** There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T_{[X]}^n$, then

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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

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$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T_n^r$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S^r_X} p(x) N(x;\mathbf{x}) > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin S^r_X$. Then

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$$- n \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (-\log p(x))$$

where $\eta > 0$ as $\eta \to 0$, proving Property 1.
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1) If $x \in T^n_{[X] \delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $x \in T^n_{[X] \delta}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x;x)} > 0$$

because $N(x;x) = 0$ for all $x \not\in S_X$. Then

$$\log p(x)$$

$$= \sum_x N(x;x) \log p(x)$$

$$= \sum_x (N(x;x) - np(x) + np(x)) \log p(x)$$

$$= n \sum_x p(x) \log p(x)$$

$$- n \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (- \log p(x)).$$
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X] \delta}$, then

$$2^{-n(H(X) + \eta)} \leq p(x) \leq 2^{-n(H(X) - \eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T^n_{[X] \delta}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$

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$$- n \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (- \log p(x))$$

where $\eta \to 0$ as $\delta \to 0$, proving Property 1.
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X]^{\delta}}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T^n_{[X]^{\delta}}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x;x)} > 0$$

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$$= n \sum_x p(x) \log p(x)$$

$$- n \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (\log p(x))$$

$$= \sum_{x \in S_X} p(x) \log p(x)$$

$$- n \sum_{x \notin S_X} \left( \frac{1}{n} N(x;x) - p(x) \right) (\log p(x))$$

$$\leq 2^{-n(H(X)+\eta)} \leq 2^{-n(H(X)-\eta)}.$$
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T_n^{[X]_\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T_n^{[X]_\delta}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$

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1) If $x \in T_{[X]}^{\delta}$, then

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Proof
1. To prove Property 1, for $x \in T_{[X]}^{\delta}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x;\mathbf{x})} > 0$$

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**Theorem 6.2 (Strong AEP)** There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X] \delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$ 

**Proof**

1. To prove Property 1, for $x \in T^n_{[X] \delta}$, we have

$$p(x) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$

because $N(x; x) = 0$ for all $x \not\in S_X$. Then

\[
\log p(x) = \sum_x N(x; x) \log p(x)
= \sum_x (N(x; x) - np(x) + np(x)) \log p(x)
= n \sum_x p(x) \log p(x)
- n \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (-\log p(x))
= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (-\log p(x)) \right].
\]

(1)
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X]}$, then

$$2^{-n(H(X) + \eta)} \leq p(x) \leq 2^{-n(H(X) - \eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T^n_{[X]}$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x;x)} > 0$$

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$$= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;x) - p(x)\right)(-\log p(x))\right].$$

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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

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Proof

1. To prove Property 1, for $x \in T^n_{[X] \delta}$, we have

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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T_{[X]}^{n}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T_{[X]}^{n}$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$

because $N(x; x) = 0$ for all $x \not\in S_X$. Then

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$$= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (- \log p(x)) \right].$$

(1)

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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(x) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $x \in T^n_{[X]\delta}$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^N(x;x) > 0$$

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(1)

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**Theorem 6.2 (Strong AEP)** There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

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**Proof**

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Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X] \delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^N(x; \mathbf{x}) > 0$$

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$$- n \sum_x \left( \frac{1}{n} N(x; \mathbf{x}) - p(x) \right) (- \log p(x))$$

$$= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x; \mathbf{x}) - p(x) \right) (- \log p(x)) \right].$$

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Proof

1. To prove Property 1, for $x \in T^n_{[X]\delta}$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x;x)} > 0$$

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$$- n \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (-\log p(x))$$

$$= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (-\log p(x)) \right].$$

(1)

Now consider

$$\sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (-\log p(x))$$

$$\leq \sum_x \frac{1}{n} N(x;x) - p(x) (-\log p(x))$$

$$\leq \delta.$$

2. Since $x \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;x) - p(x) \right| \leq \delta.$$
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{X, \delta}$, then

$$2^{-n(H(X) + \eta)} \leq p(x) \leq 2^{-n(H(X) - \eta)}.$$ 

Proof

1. To prove Property 1, for $x \in T^n_{X, \delta}$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$ 

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$$= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (- \log p(x)) \right].$$ 

(1)

2. Since $x \in T^n_{X, \delta}$,

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Now consider

$$\sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (- \log p(x))$$ 

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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^n_{[X] \delta}$, then

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**Proof**

1. To prove Property 1, for $x \in T^n_{[X] \delta}$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^{N(x; x)} > 0$$

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$$= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (- \log p(x)) \right].$$

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Proof

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Proof

1. To prove Property 1, for $x \in T^n_\delta$, we have

$$p(x) = p(x_1)p(x_2) \cdots p(x_n) = \prod_{x \in S_X} p(x)^N(x;x) > 0$$

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Proof

1. To prove Property 1, for $x \in T^X_\delta^n$, we have

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$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta.$$
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**Proof**

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\[
2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.
\]

Proof

1. To prove Property 1, for \( \mathbf{x} \in T^n_{[X] \delta} \), we have

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\]

\[
= n \sum_{x} p(x) \log p(x)
\]

\[
- n \sum_{x} \left( \frac{1}{n} N(x; \mathbf{x}) - p(x) \right) (- \log p(x))
\]

\[
= -n \left[ H(X) + \sum_{x} \left( \frac{1}{n} N(x; \mathbf{x}) - p(x) \right) (- \log p(x)) \right].
\]

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\sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| \leq \delta.
\]

Now consider

\[
\left| \sum_{x} \left( \frac{1}{n} N(x; \mathbf{x}) - p(x) \right) (- \log p(x)) \right|
\]

\[
\leq \sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| (- \log p(x))
\]

\[
\leq - \log \left( \min_{x} p(x) \right) \sum_{x} \frac{1}{n} N(x; \mathbf{x}) - p(x)
\]

\[
\leq - \delta \log \left( \min_{x} p(x) \right)
\]

\[
= \eta,
\]

where

\[
\eta = - \delta \log \left( \min_{x} p(x) \right) > 0.
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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T_{[X]}^n$, then

\[ 2^{-n(H(X) + \eta)} \leq p(x) \leq 2^{-n(H(X) - \eta)}. \]

Proof

1. To prove Property 1, for $x \in T_{[X]}^n$, we have

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= \sum_x (N(x; x) - np(x) + np(x)) \log p(x) \\
= n \sum_x p(x) \log p(x) \\
= -n \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (-\log p(x)) \\
= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x; x) - p(x) \right) (-\log p(x)) \right].
\]

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Now consider

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\leq \sum_x \left| \frac{1}{n} N(x; x) - p(x) \right| (-\log p(x)) \\
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\leq -\delta \log \left( \min_x p(x) \right) \\
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Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $x \in T^{n}_{[X] \delta}$, then

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Proof

1. To prove Property 1, for $x \in T^{n}_{[X] \delta}$, we have

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$$= -n \left[ H(X) + \sum_{x} \left( \frac{1}{n} N(x; x) - p(x) \right) \left( -\log p(x) \right) \right].$$

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\]

\[
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$$= np(x) \log p(x)$$

$$- n \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (-\log p(x))$$

$$= -n \left[ H(X) + \sum_x \left( \frac{1}{n} N(x;x) - p(x) \right) (-\log p(x)) \right].$$

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**Proof**

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Now consider

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$$\leq \sum_x \left( \frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (- \log p(x))$$

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or

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**Theorem 6.2 (Strong AEP)** There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

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\]

\[
= n \sum_x p(x) \log p(x) - n \sum_x \left( \frac{1}{n} N(x; \mathbf{x}) - p(x) \right) (- \log p(x))
\]

\[
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2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)},
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where $\eta \to 0$ as $\delta \to 0$, proving Property 1.
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For $n$ sufficiently large,

$$\Pr\{X \in T^n_{[X]\delta}\} > 1 - \delta.$$ 

Proof Idea

- By WLLN, w.p. $\to 1$ (with probability tends to 1), the empirical distribution of $X$ is close to $p(x)$, and so by definition $X$ is strongly typical.
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For $n$ sufficiently large,

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Proof
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For $n$ sufficiently large,

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Proof

1. To prove Property 2, we write
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Proof

1. To prove Property 2, we write

$$N(x; X) = \sum_{k=1}^{n} B_k(x),$$
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Proof

1. To prove Property 2, we write

$$N(x; X) = \sum_{k=1}^{n} B_k(x),$$

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$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$
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Proof

1. To prove Property 2, we write

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where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

for some $x, \eta > 0$.
Theorem 6.2 (Strong AEP) There exists \( \eta > 0 \) such that \( \eta \to 0 \) as \( \delta \to 0 \), and the following hold:

2) For \( n \) sufficiently large,

\[
\Pr\{ \mathbf{X} \in T^n_{|\mathbf{X}|\delta} \} > 1 - \delta.
\]

Proof

1. To prove Property 2, we write

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\[
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\]
**Theorem 6.2 (Strong AEP)** There exists $\eta > 0$ such that $\eta \rightarrow 0$ as $\delta \rightarrow 0$, and the following hold:

2) For $n$ sufficiently large,

$$\Pr\{X \in T^n_{[X]\delta}\} > 1 - \delta.$$

**Proof**

1. To prove Property 2, we write

$$N(x; X) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x)$, $k = 1, 2, \cdots, n$ are i.i.d. random variables with

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3. By WLLN, for any \( \delta > 0 \) and for any \( x \in \mathcal{X} \),

\[
\Pr\left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} < \frac{\delta}{|\mathcal{X}|}
\]

for \( n \) sufficiently large.

4. Then

\[
\Pr\left\{ \left| \frac{1}{n} N(x; X) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} > \frac{\delta}{|\mathcal{X}|}
\]

\[
= \Pr\left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]

\[
= \Pr\left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]

\[
\leq \sum_x \Pr\left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]

\[
< \sum_x \frac{\delta}{|\mathcal{X}|}
\]
Theorem 6.2 (Strong AEP) There exists \( \eta > 0 \) such that \( \eta \to 0 \) as \( \delta \to 0 \), and the following hold:

2) For \( n \) sufficiently large,

\[
\Pr\{X \in T^n_{|X|\delta}\} > 1 - \delta.
\]

Proof

1. To prove Property 2, we write

\[
N(x; X) = \sum_{k=1}^{n} B_k(x),
\]

where

\[
B_k(x) = \begin{cases} 
  1 & \text{if } X_k = x \\
  0 & \text{if } X_k \neq x.
\end{cases}
\]

2. Then \( B_k(x), k = 1, 2, \cdots, n \) are i.i.d. random variables with

\[
\Pr\{B_k(x) = 1\} = p(x)
\]

and

\[
\Pr\{B_k(x) = 0\} = 1 - p(x).
\]

Note that

\[
EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).
\]

3. By WLLN, for any \( \delta > 0 \) and for any \( x \in X \),

\[
\Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \right\} < \frac{\delta}{|X|}
\]

for \( n \) sufficiently large.

4. Then

\[
\Pr \left\{ \left| \frac{1}{n} N(x; X) - p(x) \right| > \frac{\delta}{|X|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \right\} \right\}
\]

\[
\leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \right\}
\]

\[
< \sum_{x} \frac{\delta}{|X|}
\]

\[
= \delta.
\]
Theorem 6.2 (Strong AEP)

There exists \( \alpha > 0 \) such that \( \alpha \not= 0 \) as \( \alpha \rightarrow 0 \), and the following hold:

1. To prove Property 2, we write

\[
N(x; \mathbf{x}) = \sum_{k=1}^{n} B_k(x) \]

where

\[
B_k(x) = \begin{cases} 
1 & \text{if } X_k = x \\
0 & \text{if } X_k \neq x 
\end{cases}
\]

2. Then \( B_k(x), k = 1, 2, \ldots, n \) are i.i.d. random variables with

\[
\Pr\{B_k(x) = 1\} = p(x) \quad \text{and} \quad \Pr\{B_k(x) = 0\} = 1 - p(x).
\]

Note that \( \mathbb{E}B_k(x) = \frac{1}{p(x)} \cdot 0 + p(x) \cdot 1 = p(x) \).

3. By WLLN, for any \( \alpha > 0 \) and for any \( x \in X \),

\[
\Pr\left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \text{ for some } x \right\}
\]

4. Then

\[
\Pr\left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \text{ for some } x \right\}
\]

\[
= \mathbb{P} \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \right\} \right\}
\]

\[
\leq \sum_{x} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|X|} \right\}
\]

\[
< \sum_{x} \frac{\delta}{|X|} = \delta.
\]
4. Then

\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]

\[
\leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]

\[
< \sum_{x} \frac{\delta}{|\mathcal{X}|}
\]

\[
= \delta.
\]
4. Then

\[ \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]

\[ \leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]

\[ < \sum_{x} \frac{\delta}{|\mathcal{X}|} \]

\[ = \delta. \]

5. Now

\[ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \]
4. Then
\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]
\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]
\[
= \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]
\[
\leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]
\[
< \sum_{x} \frac{\delta}{|\mathcal{X}|}
\]
\[
= \delta.
\]

5. Now
\[
\sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta
\]
implies
\[
\left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
\]
4. Then

\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]

\[
\leq \sum_x \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]

\[
< \sum_x \frac{\delta}{|\mathcal{X}|}
\]

\[
= \delta.
\]

5. Now

\[
\sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta
\]

implies

\[
\left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
\]
4. Then
\[ \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]
\[ = \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]
\[ = \Pr \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]
\[ \leq \sum_x \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]
\[ < \sum_x \frac{\delta}{|\mathcal{X}|} \]
\[ = \delta. \]

5. Now
\[ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \]
implies
\[ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \]
4. Then

\[ \text{Pr} \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \text{Pr} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \text{Pr} \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]

\[ \leq \sum_x \text{Pr} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]

\[ < \sum_x \frac{\delta}{|\mathcal{X}|} \]

\[ = \delta. \]

5. Now

\[ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \]

implies

\[ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}. \]

Then we have

\[ \text{Pr} \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\} \]
4. Then
\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
= \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
< \sum_{x} \frac{\delta}{|\mathcal{X}|}
= \delta.
\]

5. Now
\[
\sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta
\]
implies
\[
\left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
\]
Then we have
\[
\Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\}
\leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}.
\]
4. Then

\[ \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]

\[ \leq \sum_x \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]

\[ < \sum_x \frac{\delta}{|\mathcal{X}|} \]

\[ = \delta. \]

5. Now

\[ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \]

implies

\[ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}. \]

Then we have

\[ \Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\} \]

\[ \leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}. \]

Then

\[ \Pr \left\{ \mathbf{x} \in T_n^{\delta} \right\} \]
4. Then

\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]

\[
= \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]

\[
\leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]

\[
< \sum_{x} \frac{\delta}{|\mathcal{X}|}
\]

\[
= \delta.
\]

5. Now

\[
\sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta
\]

implies

\[
\left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
\]

Then we have

\[
\Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\}
\]

\[
\leq \Pr \left\{ \frac{1}{n} N(x; \mathbf{x}) - p(x) > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}.
\]

Then

\[
\Pr \left\{ \mathbf{x} \in T_{[X]}^{n/\delta} \right\}
\]

\[
= \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| \leq \delta \right\}
\]
4. Then
\[ \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]
\[ = \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]
\[ = \Pr \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]
\[ \leq \sum_x \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]
\[ < \sum_x \frac{\delta}{|\mathcal{X}|} \]
\[ = \frac{\delta}{|\mathcal{X}|} \]
\[ = \delta. \]

5. Now
\[ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \]
implies
\[ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}. \]

Then we have
\[ \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \right\} \]
\[ \leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\} . \]

Then
\[ \Pr \left\{ \mathbf{X} \in T_{|\mathcal{X}|}^n \delta \right\} \]
\[ = \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| \leq \delta \right\} \]
\[ = 1 - \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \right\} \]
4. Then
\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]
\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]
\[
= \Pr \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]
\[
\leq \sum_x \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]
\[
< \sum_x \frac{\delta}{|\mathcal{X}|}
\]
\[
= \delta.
\]

5. Now
\[
\sum_x \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta
\]
implies
\[
\left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
\]
Then we have
\[
\Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\}
\]
\[
\leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}.
\]
Then
\[
\Pr \left\{ \mathbf{x} \in T_{[\mathcal{X}]\delta} \right\}
\]
\[
= \Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| \leq \delta \right\}
\]
\[
= 1 - \Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \right\}.
4. Then
\[
\Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]
\[= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\}
\]
\[= \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\}
\]
\[\leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\}
\]
\[< \sum_{x} \frac{\delta}{|\mathcal{X}|}
\]
\[= \delta.
\]
5. Now
\[
\sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta
\]
implies
\[
\left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
\]
Then we have
\[
\Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\}
\]
\[\leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}.
\]
Then
\[
\Pr \left\{ \mathbf{x} \in T_{[\mathcal{X}]\delta} \right\}
\]
\[= \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| \leq \delta \right\}
\]
\[= 1 - \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \right\}
\]
\[\geq 1 - \Pr \left\{ \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}.
4. Then

\[ \Pr \left\{ \left| \frac{1}{n} N(x; X) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]

\[ \leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]

\[ \leq \sum_{x} \frac{\delta}{|\mathcal{X}|} \]

\[ = \delta \]

5. Now

\[ \sum_{x} \left| \frac{1}{n} N(x; x) - p(x) \right| > \delta \]

implies

\[ \left| \frac{1}{n} N(x; x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}. \]

Then we have

\[ \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; x) - p(x) \right| > \delta \right\} \]

\[ \leq \Pr \left\{ \left| \frac{1}{n} N(x; x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}. \]

Then

\[ \Pr \left\{ X \in T_{|\mathcal{X}|}\delta \right\} \]

\[ = \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; X) - p(x) \right| \leq \delta \right\} \]

\[ = 1 - \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; X) - p(x) \right| > \delta \right\} \]

\[ \geq 1 - \Pr \left\{ \left| \frac{1}{n} N(x; X) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\}. \]
4. Then

\[ \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \right\} \]

\[ = \Pr \left\{ \bigcup_{x} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \]

\[ \leq \sum_{x} \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \]

\[ \leq \sum_{x} \frac{\delta}{|\mathcal{X}|} \]

\[ = \delta \]

5. Now

\[ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \]

implies

\[ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \]

Then we have

\[ \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\} \]

\[ \leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\} \]

Then

\[ \Pr \left\{ \mathbf{x} \in T_{[|\mathcal{X}|]\delta} \right\} \]

\[ = \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| \leq \delta \right\} \]

\[ = 1 - \Pr \left\{ \sum_{x} \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \right\} \]

\[ \geq 1 - \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X} \right\} \]

\[ > 1 - \delta, \]
4. Then

\[ \Pr \left\{ \frac{1}{n} N(x; \mathbf{X}) - p(x) > \frac{\delta}{|\mathcal{X}|} \right\} \]

\[
= \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \\
= \Pr \left\{ \bigcup_x \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \right\} \\
\leq \sum_x \Pr \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \\
< \sum_x \frac{\delta}{|\mathcal{X}|} \\
= \delta.
\]

5. Now

\[ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \]

implies

\[ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \]

for some \( x \in \mathcal{X} \).

Then we have

\[ \Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \delta \right\} \]

\[
\leq \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \\
< \sum_x \frac{\delta}{|\mathcal{X}|} \\
= \delta.
\]

Then

\[ \Pr \{ \mathbf{X} \in T_{[\mathcal{X}]\delta} \} \]

\[
= \Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| \leq \delta \right\} \\
= 1 - \Pr \left\{ \sum_x \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \delta \right\} \\
\geq 1 - \Pr \left\{ \left| \frac{1}{n} N(x; \mathbf{X}) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} \\
> 1 - \delta,
\]

proving Property 2.
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

3) For $n$ sufficiently large,

$$(1-\delta)2^n(H(X)-\eta) \leq |T^X_n| \leq 2^n(H(X)+\eta).$$
Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

3) For $n$ sufficiently large,

$$(1-\delta)2^n(H(X)\eta) \leq |T^n[X]_\delta| \leq 2^n(H(X)+\eta).$$

Proof Follows from Property 1 and Property 2 in exactly the same way as in Theorem 5.3. (Exercise)
Theorem 6.2 (Strong AEP) There exists \( \eta > 0 \) such that \( \eta \to 0 \) as \( \delta \to 0 \), and the following hold:

3) For \( n \) sufficiently large,

\[
(1-\delta)2^{n(H(X)-\eta)} \leq |T^n_{[X]_\delta}| \leq 2^{n(H(X)+\eta)}.
\]

Proof Follows from Property 1 and Property 2 in exactly the same way as in Theorem 5.3. (Exercise)

---

Theorem 5.3 (Weak AEP II)

1) If \( x \in W^n_{[X]_\epsilon} \), then

\[
2^{-n(H(X)+\epsilon)} \leq p(x) \leq 2^{-n(H(X)-\epsilon)}.
\]

2) For \( n \) sufficiently large,

\[
\Pr\{X \in W^n_{[X]_\epsilon}\} > 1 - \epsilon.
\]

3) For \( n \) sufficiently large,

\[
(1 - \epsilon)2^{n(H(X)-\epsilon)} \leq |W^n_{[X]_\epsilon}| \leq 2^{n(H(X)+\epsilon)}.
\]
Theorem 6.3  For sufficiently large $n$, there exists $\varphi(\delta) > 0$ such that

$$\Pr\{X \not\in T^n_{[X]_\delta}\} < 2^{-n\varphi(\delta)}.$$ 

Proof  Chernoff bound.